WELL-POSEDNESS

 Sébastien Boisgérault

CONTROL ENGINEERING WITH PYTHON

- 📖 Course Materials
-
- 🏛️ ITN, Mines Paris - PSL University

SYMBOLS

🐍	Code	🔍	Worked Example
📈	Graph	🧩	Exercise
🏷️	Definition	💻	Numerical Method
💎	Theorem	🧮	Analytical Method
📝	Remark	🧠	Theory
ℹ	Information	🗝️	Hint
⚠️	Warning	🔓	Solution

🐍 IMPORTS

```
from numpy import *
from numpy.linalg import *
from scipy.integrate import solve_ivp
from matplotlib.pyplot import *
```

🐍 STREAM PLOT HELPER

```
def Q(f, xs, ys):
    X, Y = meshgrid(xs, ys)
    fx = vectorize(lambda x, y: f([x, y])[0])
    fy = vectorize(lambda x, y: f([x, y])[1])
    return X, Y, fx(X, Y), fy(X, Y)
```

🏷️ WELL-POSEDNESS

Make sure that a system is “sane” (not “pathological”):

Well-Posedness:

- **Existence** +
- **Uniqueness** +
- **Continuity**.

We will define and study each one in the sequel.

LOCAL VS GLOBAL

So far, we have mostly dealt with **global** solutions $x(t)$ of IVPs, defined for any $t \geq t_0$.

This concept is sometimes too stringent.

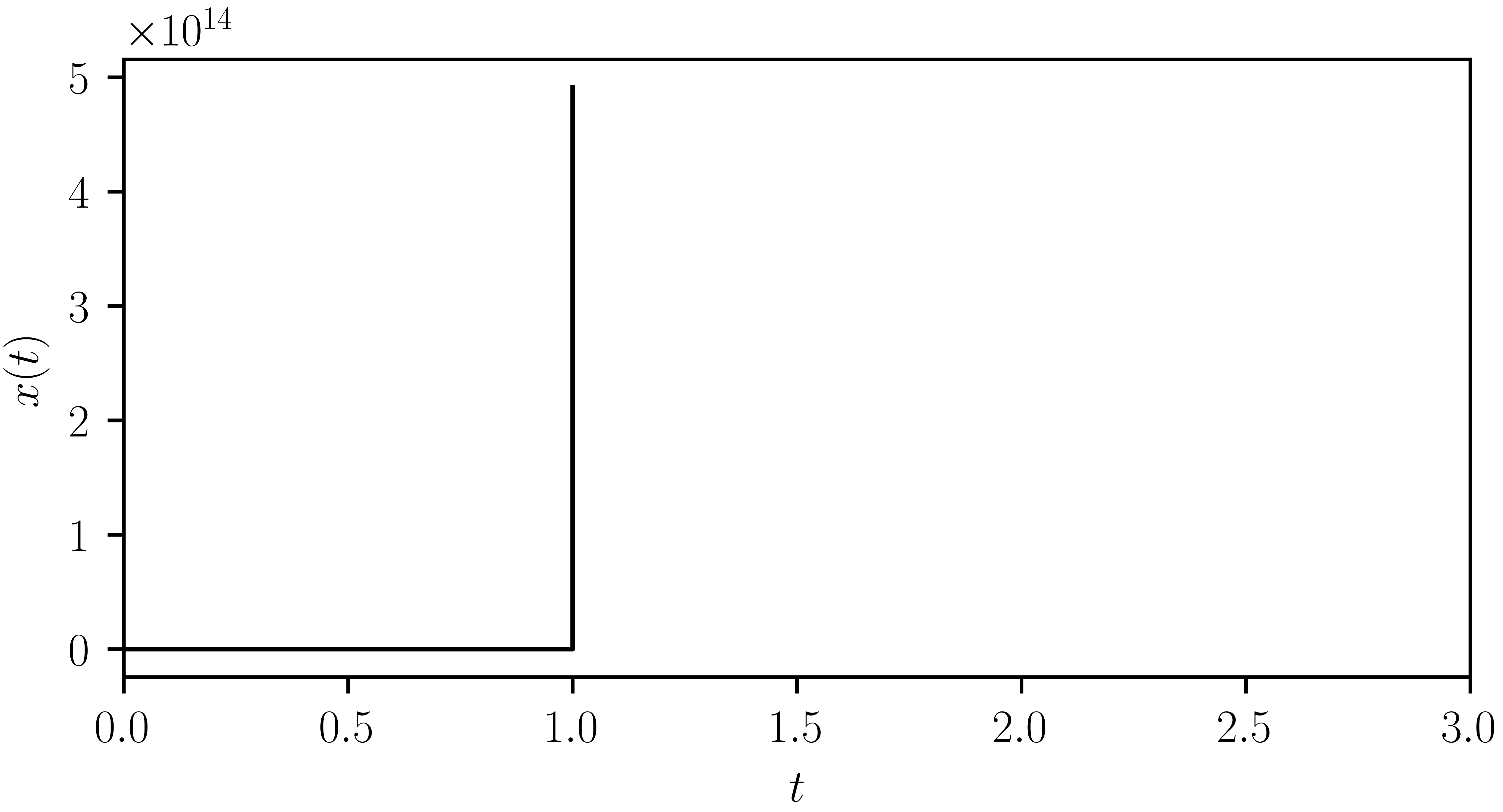
🔍 FINITE-TIME BLOW-UP

Consider the IVP

$$\dot{x} = x^2,\ x(0) = 1.$$

🐍 💻 📈

```
def fun(t, y):
    return y * y
t0, tf, y0 = 0.0, 3.0, array([1.0])
result = solve_ivp(fun, t_span=[t0, tf], y0=y0)
figure()
plot(result["t"], result["y"][0], "k")
xlim(t0, tf); xlabel("$t$"); ylabel("$x(t)$")
```


$\times 10^{14}$
5
4
3
2
1
0
$x(t)$
0.0
0.5
1.0
1.5
2.0
2.5
3.0
t

LOCAL VS GLOBAL

🤕 Ouch.

There is actually no **global** solution.

However there is a **local** solution $x(t)$,

- defined for $t \in [t_0, \tau[$
- for **some** $\tau > t_0$.

Indeed, the function $x(t) := \dfrac{1}{1-t}$ satisfies

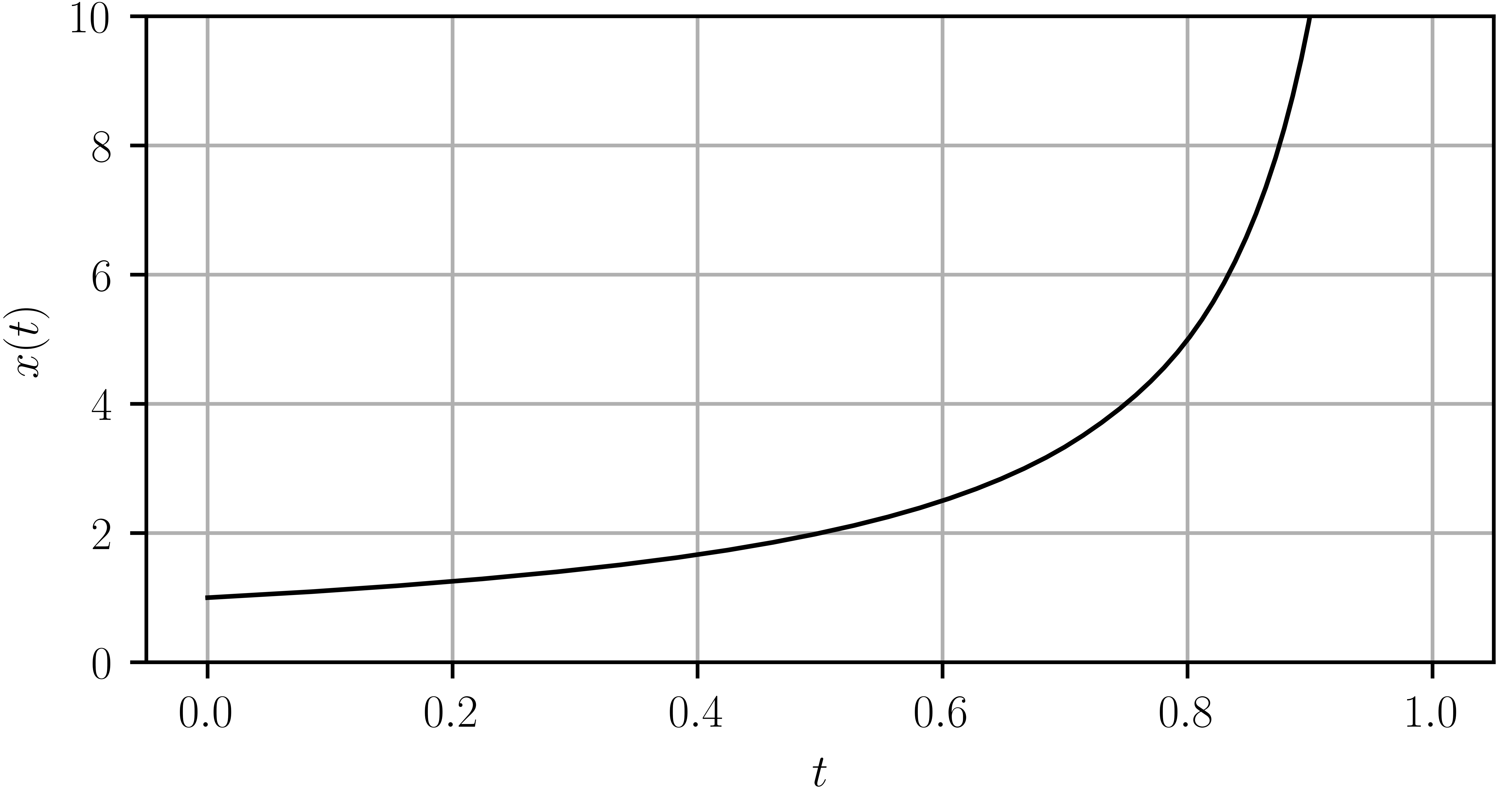
$$\dot{x}(t) = \frac{d}{dt}x(t) = -\frac{-1}{(1-t)^2} = (x(t))^2$$

and $x(0) = 1$.

⚠️ But it's defined (continuously) only for $t < 1$.

🐍 💻 📈

```
tf = 1.0
r = solve_ivp(fun, [t0, tf], y0,
              dense_output=True)
figure()
t = linspace(t0, tf, 1000)
plot(t, r["sol"](t)[0], "k")
ylim(0.0, 10.0); grid();
xlabel("$t$"); ylabel("$x(t)$")
```



10
8
6
4
2
0
$x(t)$
0.0
0.2
0.4
0.6
0.8
1.0
t

This local solution is also **maximal**:

You cannot extend this solution beyond $\tau = 1.0$.

🏷️ LOCAL SOLUTION

A solution $x : I \to \mathbb{R}^n$ of the IVP

$$\dot{x} = f(x), \; x(t_0) = x_0$$

is (forward and) **local** if $I = [t_0, \tau[$ for some τ such that $t_0 < \tau \leq +\infty$.

GLOBAL SOLUTION

A solution $x : I \to \mathbb{R}^n$ of the IVP

$$\dot{x} = f(x), \; x(t_0) = x_0$$

is (forward and) **global** if $I = [t_0, +\infty[$.

🏷️ MAXIMAL SOLUTION

A (local) solution $x : [0, \tau[$ to an IVP is **maximal** if there is no other solution

- defined on $[0, \tau'[$ with $\tau' > \tau$,
- whose restriction to $[0, \tau[$ is x.

🧩 MAXIMAL SOLUTIONS

Consider the IVP

$$\dot{x} = x^2,\ x(0) = x_0 \neq 0.$$

1. 🧮

Find a closed-formed local solution $x(t)$ of the IVP.

🗝️ **Hint:** assume that $x(t) \neq 0$ then compute

$$\frac{d}{dt}\frac{1}{x(t)}.$$

2. 🧠

Make sure that your solutions are maximal.

🔓 MAXIMAL SOLUTIONS

1. 🔓

As long as $x(t) \neq 0$,

$$\frac{d}{dt}\frac{1}{x(t)} = -\frac{\dot{x}(t)}{x(t)^2} = 1.$$

By integration, this leads to

$$\frac{1}{x(t)} - \frac{1}{x_0} = -t$$

and thus provides

$$x(t) = \frac{1}{\frac{1}{x_0} - t} = \frac{x_0}{1 - x_0 t}.$$

which is indeed a solution as long as the denominator is not zero.

2. 🔓

- If $x_0 < 0$, this solution is valid for all $t \geq 0$ and thus maximal.
- If $x_0 > 0$, the solution is defined until $t = 1/x(0)$ where it blows up. Thus, this solution is also maximal.

🙁 BAD NEWS (1/3)

Sometimes things get worse than simply having no global solution.

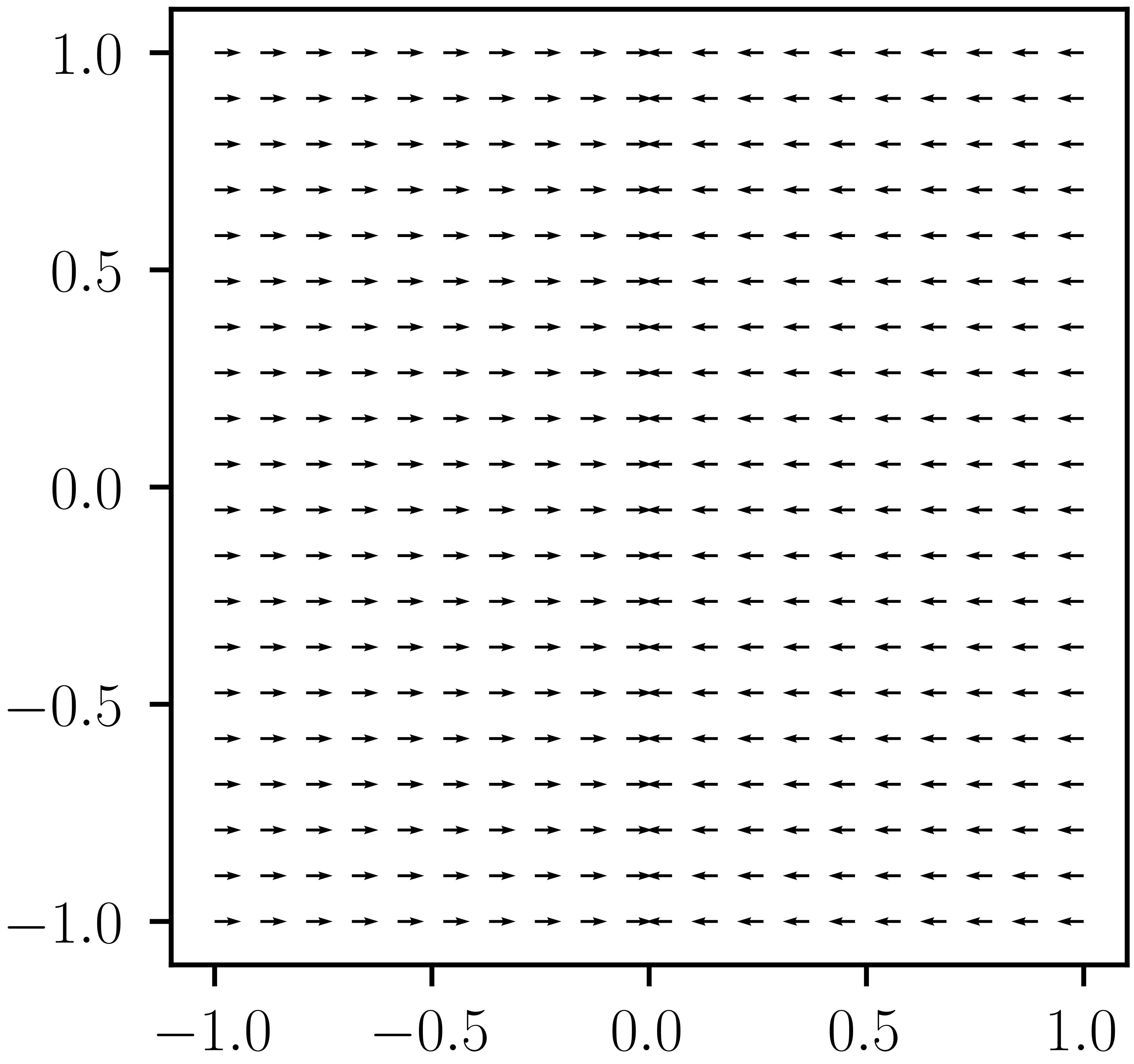
🔍 NO LOCAL SOLUTION

Consider the scalar IVP with initial value $x(0) = (0, 0)$ and right-hand side

$$f(x_1, x_2) = \begin{cases} (+1, 0) & \text{if } \; x_1 < 0 \\ (-1, 0) & \text{if } \; x_1 \geq 0. \end{cases}$$

🐍 📈 NO LOCAL SOLUTION

```python
def f(x1x2):
    x1, x2 = x1x2
    dx1 = 1.0 if x1 < 0.0 else -1.0
    return array([dx1, 0.0])
figure()
x1 = x2 = linspace(-1.0, 1.0, 20)
gca().set_aspect(1.0)
quiver(*Q(f, x1, x2), color="k")
```


1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0

💎 NO LOCAL SOLUTION

This system has no solution, not even a local one, when $x(0) = (0, 0)$.

🧠 PROOF

- Assume that $x : [0, \tau[\to \mathbb{R}$ is a local solution.
- Since $\dot{x}(0) = -1 < 0$, for some small enough $0 < \epsilon < \tau$ and any $t \in]0, \epsilon]$, we have $x(t) < 0$.
- Consequently, $\dot{x}(t) = +1$ and thus by integration

$$x(\epsilon) = x(0) + \int_0^\epsilon \dot{x}(t)\, dt = \epsilon > 0,$$

which is a contradiction.

🙂 GOOD NEWS (1/3)

However, a local solution exists under very mild assumptions.

💎 EXISTENCE

If f is continuous,

- There is a (at least one) **local** solution to the IVP

  $\dot{x} = f(x)$ and $x(t_0) = x_0$.

- Any local solution on some $[t_0, \tau[$ can be extended to a (at least one) **maximal** one on some $[t_0, t_\infty[$.

📝 **Note:** a maximal solution is **global** iff $t_\infty = +\infty$.

💎 MAXIMAL SOLUTIONS

A solution on $[t_0, \tau[$ is maximal if and only if either

- $\tau = +\infty$: the solution is global, or
- $\tau < +\infty$ and $\lim_{t \to \tau} \|x(t)\| = +\infty$.

In plain words : a non-global solution cannot be extended further in time if and only if it “blows up”.

💎 COROLLARY

Let's assume that a local maximal solution exists.

You wonder if this solution is defined in $[t_0, t_f[$ or blows up before t_f.

For example, you wonder if a solution is global (if $t_f = +\infty$ or $t_f < +\infty$.)

🧠 PROVE EXISTENCE

Task. Show that any solution which defined on some sub-interval $[t_0, \tau]$ with $\tau < t_f$ would is bounded.

Then, no solution can be maximal on any such $[0, \tau[$ (since it doesn't blow up !). Since a maximal solution does exist, its domain is $[0, t_\infty[$ with $t_\infty \geq t_f$.

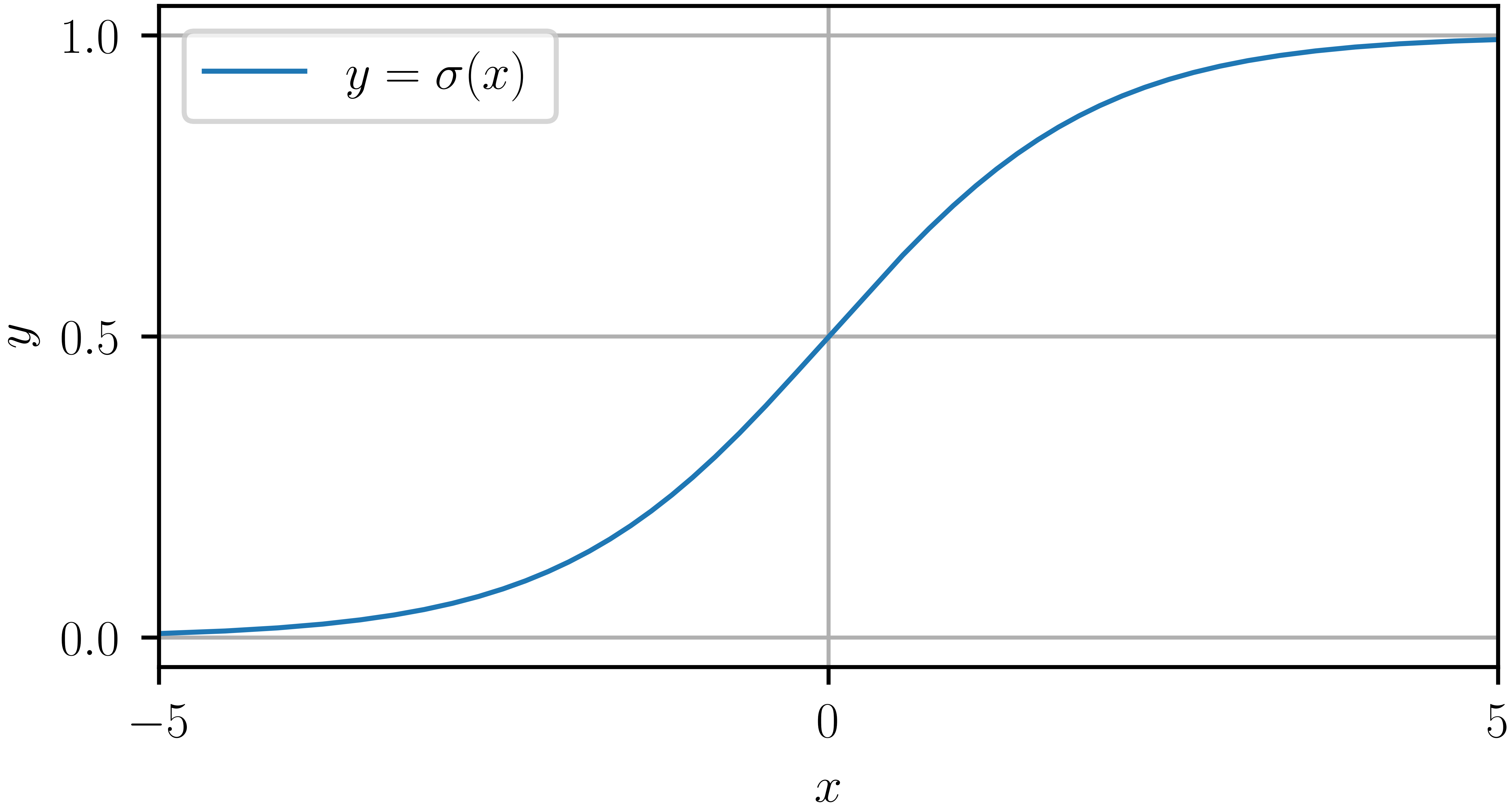
$\Rightarrow$ **a solution is defined on** $[t_0, t_f[$**.**

🧩 SIGMOID

Consider the dynamical system

$$\dot{x} = \sigma(x) := \frac{1}{1 + e^{-x}}.$$

```python
def sigma(x):
  return 1 / (1 + exp(-x))
figure()
x = linspace(-7.0, 7.0, 1000)
plot(x, sigma(x), label="$y=\sigma(x)$")
grid(True)
```



1.0
0.5
0.0
y
-5
0
5
x
$y = \sigma(x)$

1. 🧮 EXISTENCE

Show that there is a (at least one) maximal solution to each initial condition.

2. 🧮 GLOBAL

Show that any such solution is global.

🔓 SIGMOID

1. 🔓 EXISTENCE

The sigmoid function σ is continuous.

Consequently, 💎 Existence proves the existence of a (at least one) maximal solution.

2. 🔓 GLOBAL

Let $x : [0, \tau[\to \mathbb{R}$ be a maximal solution to the IVP. We have

$$0 \leq \dot{x}(t) = \sigma(x(t)) \leq 1, \; 0 \leq t < \tau$$

and by integration,

$$|x(t)| \leq |x(0)| + t$$

Thus, it cannot blow-up in finite time; by 💎 Maximal Solutions, it is global.

🧩 PENDULUM

Consider the pendulum, subject to a torque c

$$ml^2\ddot{\theta} + b\dot{\theta} + mg\ell\sin\theta = c(\theta, \dot{\theta})$$

We assume that the torque provides a bounded power:

$$P := c(\theta, \dot{\theta})\dot{\theta} \leq P_M < +\infty.$$

1. 🧮

Show that for any initial state, there is a global solution $(\theta, \dot{\theta})$.

🗝️ **Hint.** Compute the derivative with respect to t of

$$E = \frac{1}{2} m\ell^2\dot{\theta}^2 - mg\ell\cos\theta.$$

🔓 PENDULUM

1. 🔓

Since the system vector field

$$(\theta, \dot{\theta}) \to \left(\dot{\theta}, (-b/m\ell^2)\dot{\theta} - (g/\ell)\sin\theta + c(\theta, \dot{\theta})/m\ell^2\right)$$

is continuous, 💎 Existence yields the existence of a (at least one) maximal solution.

Additionally,

$$\begin{aligned}\dot{E} &= \frac{d}{dt}\left(\frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta\right) \\ &= -b\dot{\theta}^2 + c(\theta,\dot{\theta})\dot{\theta} \\ &\leq P_M < +\infty.\end{aligned}$$

By integration

$$E(t) = \frac{1}{2}m\ell^2\dot{\theta}^2(t) - mg\ell\cos\theta(t) \leq E(0) + P_M t$$

Hence, since $|\cos\theta(t)| \leq 1$,

$$|\dot{\theta}(t)| \leq \sqrt{\frac{2E(0)}{m\ell^2} + \frac{2g}{\ell} + \frac{2P_M}{m\ell^2}t}$$

Thus, $\dot{\theta}(t)$ cannot blow-up in finite time. Since

$$|\theta(t)| \leq |\theta(0)| + \int_0^t |\dot{\theta}(s)| \, ds,$$

$\theta(t)$ cannot blow-up in finite time either.

By 💎 Maximal Solutions, any maximal solution is global.

LINEAR SYSTEMS

Let $A \in \mathbb{R}^{n \times n}$.

Consider the dynamical system

$$\dot{x} = Ax, \; x \in \mathbb{R}^n.$$

1. 🧮

Show that

$$y(t) := \|x(t)\|^2$$

is differentiable and satisfies

$$\dot{y}(t) \leq 2\alpha y(t)$$

for some $\alpha \geq 0$. 🔓

2. 🧮

Let

$$z(t) := y(t)e^{-2\alpha t}.$$

Compute $\dot{z}(t)$ and deduce that

$$0 \leq y(t) \leq y(0)e^{2\alpha t}.$$

3. 🧮

Prove that for any initial state $x(0) \in \mathbb{R}^n$ there is a corresponding global solution $x(t)$. 🔓

🔓 LINEAR SYSTEMS

1. 🔓

By definition of $y(t)$ and since $\dot{x}(t) = Ax(t)$,

$$\begin{aligned}\dot{y}(t) &= \frac{d}{dt}\|x(t)\|^2 \\ &= \frac{d}{dt}x(t)^t x(t) \\ &= \dot{x}(t)^t x(t) + x(t)^t \dot{x}(t) \\ &= x(t)^t A^t x(t) + x(t)^t A x(t).\end{aligned}$$

Let α denote the largest singular value of A (i.e. the operator norm $\|A\|$).

$$\alpha := \sigma_{\max}(A) = \|A\|.$$

For any vector $u \in \mathbb{R}^n$, we have

$$\|Au\| \leq \|A\|\|u\|.$$

By the triangle inequality and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\dot{y}(t) &= \|x(t)^t A^t x(t) + x(t)^t A x(t)\| \\ &\leq \|(Ax(t))^t x(t)\| + \|x(t)^t (Ax(t))\| \\ &\leq \|Ax(t)\|\|x(t)\| + \|x(t)\|\|Ax(t)\| \\ &\leq \|A\|\|x(t)\|\|x(t)\| + \|x(t)\|\|A\|\|x(t)\| \\ &= 2\|A\|y(t)\end{aligned}$$

and thus $\dot{y}(t) \leq 2\alpha y(t)$ with $\alpha := \|A\|$.

2. 🔓

Since $y(t) = \|x(t)\|^2$, the inequality $0 \le y(t)$ is clear.

Since $z(t) = y(t)e^{-2\alpha t}$,

$$\begin{aligned}
\dot{z}(t) &= \frac{d}{dt} y(t)e^{-2\alpha t} \\
&= \dot{y}(t)e^{-2\alpha t} + y(t)(-2\alpha e^{-\alpha t}) \\
&= (\dot{y}(t) - 2\alpha y(t))e^{-2\alpha t} \\
&\le 0.
\end{aligned}$$

By integration

$$y(t)e^{-2\alpha t} = z(t) = z(0) + \int_0^t \dot{z}(s)\,ds$$
$$\leq z(0) = y(0),$$

hence

$$y(t) \leq y(0)e^{2\alpha t}.$$

3. 🔓

The vector field

$$x \in \mathbb{R}^n \to Ax$$

is continuous, thus by 💎 Existence there is a maximal solution $x : [0, t_\infty[$ for any initial state $x(0)$.

Moreover,

$$\|x(t)\| = \sqrt{\|y(t)\|} \leq \sqrt{y(0)e^{2\alpha t}} = \|x(0)\|e^{\alpha t}.$$

Hence there is no finite-time blow-up and the maximal solution is global.

🏷️ UNIQUENESS

In the current context, **uniqueness** means uniqueness of the maximal solution to an IVP.

🙁 BAD NEWS (2/3)

Uniqueness of solutions, even the maximal ones, is not granted either.

🔍 NON-UNIQUENESS

The IVP

$$\dot{x} = \sqrt{x},\ x(0) = 0$$

has several maximal (global) solutions.

PROOF

For any $\tau \geq 0$, x_τ is a solution:

$$x_\tau(t) = \begin{vmatrix} 0 & \text{if } t \leq \tau, \\ 1/4 \times (t-\tau)^2 & \text{if } t > \tau. \end{vmatrix}$$

🙂 GOOD NEWS (2/3)

However, uniqueness of maximal solution holds under mild assumptions.

🏷️ JACOBIAN MATRIX

$$x = (x_1, \ldots, x_n),\ \ f(x) = (f_1(x), \ldots, f_n(x)).$$

Jacobian matrix of f:

$$\frac{\partial f}{\partial x} := \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

💎 UNIQUENESS

If $\partial f / \partial x$ exists and is continuous, the maximal solution is unique.

🙁 BAD NEWS (3/3)

An infinitely small error in the initial value could result in a finite error in the solution, even in finite time.

That would severely undermine the utility of any approximation method.

🏷️ CONTINUITY

Instead of denoting $x(t)$ the solution, use $x(t, x_0)$ to emphasize the dependency w.r.t. the initial state.

Continuity w.r.t. the initial state means that if $x(t, x_0)$ is defined on $[t_0, \tau]$ and $t \in [t_0, \tau]$:

$$x(t, y) \to x(t, x_0) \text{ when } y \to x_0$$

and that this convergence is uniform w.r.t. t.

🙂 GOOD NEWS (3/3)

However, continuity w.r.t. the initial value holds under mild assumptions.

💎 CONTINUITY

Assume that $\partial f / \partial x$ exists and is continuous.

Then the dynamical system is continous w.r.t. the initial state.

🔍 PREY-PREDATOR

Let

$$\begin{aligned}\dot{x} &= \alpha x - \beta x y \\ \dot{y} &= \delta x y - \gamma y\end{aligned}$$

with $\alpha = 2/3, \beta = 4/3, \delta = \gamma = 1.0.$

```python
alpha = 2 / 3; beta = 4 / 3; delta = gamma = 1.0

def fun(t, y):
    x, y = y
    u = alpha * x - beta * x * y
    v = delta * x * y - gamma * y
    return array([u, v])
```

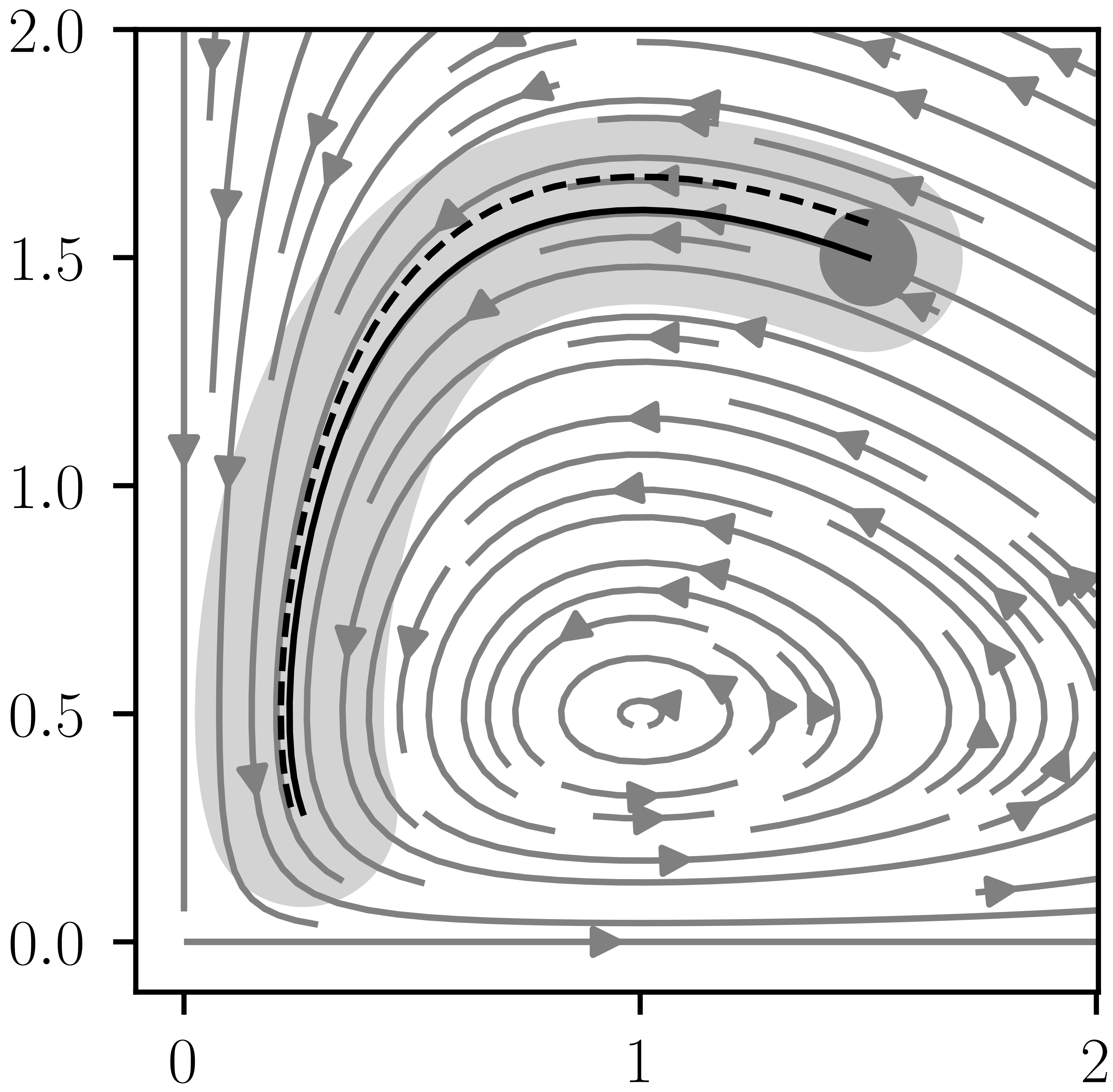
```python
tf = 3.0
result = solve_ivp(
  fun,
  t_span=(0.0, tf),
  y0=[1.5, 1.5],
  max_step=0.01)
x, y = result["y"][0], result["y"][1]
```

```python
def display_streamplot():
    ax = gca()
    xr = yr = linspace(0.0, 2.0, 1000)
    def f(y):
        return fun(0, y)
    streamplot(*Q(f, xr, yr), color="grey")
```

```python
def display_reference_solution():
    for xy in zip(x, y):
        x_, y_ = xy
        gca().add_artist(Circle((x_, y_),
                                0.2, color="#d3d3d3"))
    gca().add_artist(Circle((x[0], y[0]), 0.1,
                            color="#808080"))
    plot(x, y, "k")
```

```python
def display_alternate_solution():
    result = solve_ivp(fun,
                       t_span=[0.0, tf],
                       y0=[1.5, 1.575],
                       max_step=0.01)
    x, y = result["y"][0], result["y"][1]
    plot(x, y, "k--")
```

```
figure()
display_streamplot()
display_reference_solution()
display_alternate_solution()
axis([0,2,0,2]); axis("square")
```


2.0
1.5
1.0
0.5
0.0
0
1
2

🧩 CONTINUITY

Let $h \geq 0$ and $x^h(t)$ be the solution of the IVP

$$\dot{x} = x,\ x^h(0) = 1 + h.$$

1. 🧮

Let $\epsilon > 0$ and $\tau \geq 0$.

Find the largest $\delta > 0$ such that $|h| < \delta$ ensures that

$$\text{for any } t \in [t_0, \tau], |x^h(t) - x^0(t)| \leq \epsilon$$

2. 🧮

What is the behavior of δ when τ goes to infinity?

🔓 CONTINUITY

2. 🔓

The solution $x^h(t)$ to the IVP is

$$x^h(t) = (1 + h)e^t.$$

Hence,

$$|x^h(t) - x^0(t)| = |(1 + h)e^t - e^t| = |h|e^t$$

$$\max_{t \in [0,\tau]} |x^h(t) - x^0(t)| = |h|e^\tau.$$

Thus, the smallest δ such that $|h| \leq \delta$ yields

$$\max_{t \in [0,\tau]} |x^h(t) - x^0(t)| \leq \epsilon.$$

is $\delta = \varepsilon e^{-\tau}$.

2. 🔓

For any $\varepsilon > 0$,

$$\lim_{\tau \to +\infty} \delta = 0.$$

🧩 CONTINUITY ISSUES

Consider the IVP

$$\dot{x} = \sqrt{|x|},\ x(0) = x_0 \in \mathbb{R}.$$

1. 💻 📈

Solve numerically this IVP for $t \in [0, 1]$ and $x_0 = 0$ and plot the result.

Then, solve it again for $x_0 = 0.1$, $x_0 = 0.01$, etc. and plot the results.

2. 🔬

Does the solution seem to be continuous with respect to the initial value?

3. 🧠

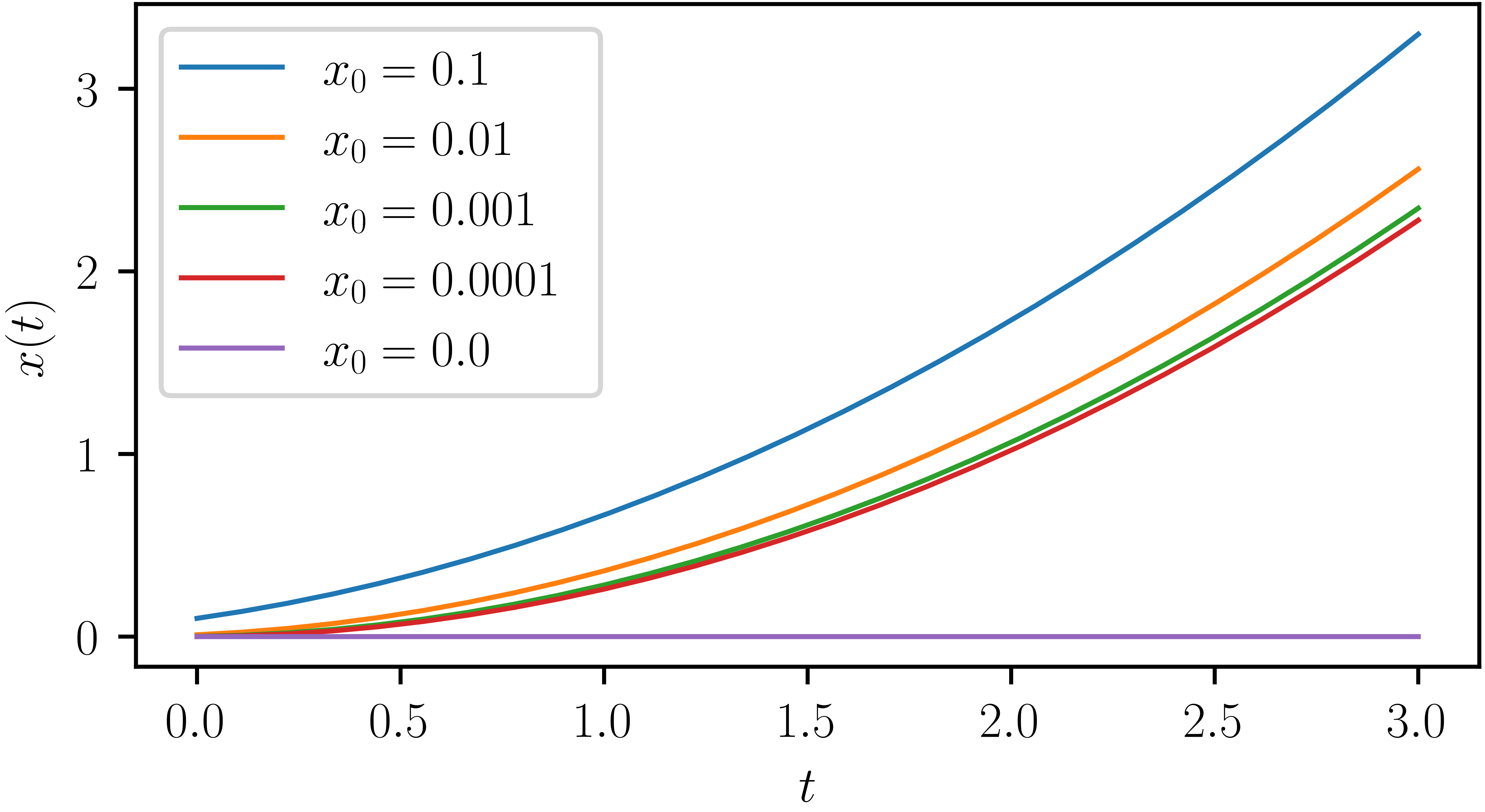
Explain this experimental result.

🔓 CONTINUITY ISSUES

1. 🔓

```
def fun(t, y):
  x = y[0]
  dx = sqrt(abs(y))
  return [dx]
tspan = [0.0, 3.0]
t = linspace(tspan[0], tspan[1], 1000)
```

```python
figure()
for x0 in [0.1, 0.01, 0.001, 0.0001, 0.0]:
    r = solve_ivp(fun, tspan, [x0],
        dense_output=True)
    plot(t, r["sol"](t)[0],
         label=f"$x_0 = {x0}$")
xlabel("$t$"); ylabel("$x(t)$")
legend()
```


$x_0 = 0.1$
$x_0 = 0.01$
$x_0 = 0.001$
$x_0 = 0.0001$
$x_0 = 0.0$
$x(t)$
t
0
1
2
3
0.0
0.5
1.0
1.5
2.0
2.5
3.0

2. 🔓

The solution does not seem to be continuous with respect to the initial value since the graph of the solution seems to have a limit when $x_0 \to 0^+$, but this limit is different from $x(t) = 0$ which is the numerical solution when $x_0 = 0$.

3. 🔓

The jacobian matrix of the vector field is not defined when $x = 0$, thus the continuity was not guaranted to begin with. Actually, uniqueness of the solution does not even hold here, see 🔍 Non-Uniqueness. The function $x(t) = 0$ is valid when $x_0 = 0$, but so is

$$x(t) = \frac{1}{4} t^2$$

and the numerical solution seems to converge to the second one when $x_0 \to 0^+$.

🧩 PREY-PREDATOR

Consider the system

$$\begin{aligned} \dot{x} &= \alpha x - \beta xy \\ \dot{y} &= \delta xy - \gamma y \end{aligned}$$

where α, β, δ and γ are positive.

1. 🧮

Prove that the system is well-posed.

2. 🧮 🧠

Prove that all maximal solutions such that $x(0) > 0$ and $y(0) > 0$ are global and satify $x(t) > 0$ and $y(t) > 0$ for every $t \geq 0$.

Hint 🗝️. Compute the ODE satisfied by $u = \ln x$ and $v = \ln y$ and then the derivative w.r.t. time of

$$V := \delta e^u - \gamma u + \beta e^v - \alpha v.$$

🔓 PREY-PREDATOR

🔓 1.

The jacobian matrix of the system vector field

$$f(x, y) = (\alpha x - \beta x y, \delta x y - \gamma y)$$

is defined and continuous:

$$\frac{\partial f}{\partial (x, y)} = \begin{bmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{bmatrix}$$

thus the sytem is well-posed.

🔓 2.

The (continuously differentiable) change of variable

$$F : (x, y) \mapsto (u, v) := (\ln x, \ln y)$$

is a bijection between $]0, +\infty[^2$ and $\mathbb{R}^2$.

Since

$$\frac{d}{dt}\ln x = \frac{\dot{x}}{x}, \quad \frac{d}{dt}\ln y = \frac{\dot{y}}{y}$$

the prey-predator ODE is equivalent to

$$\begin{aligned} \dot{u} &= \alpha - \beta e^{v} \\ \dot{v} &= \delta e^{u} - \gamma \end{aligned}$$

Accordingly,

$$\begin{aligned}\frac{d}{dt}V &= \delta e^u\dot{u} - \gamma\dot{u} + \beta e^v\dot{v} - \alpha\dot{v} \\ &= (\delta e^u - \gamma)\dot{u} + (\beta e^v - \alpha\dot{v}) \\ &= (\delta e^u - \gamma)(\alpha - \beta e^v) + (\beta e^v - \alpha)(\delta e^u - \gamma) \\ &= 0\end{aligned}$$

Therefore $V(u(t), v(t))$ is constant.

Now, the function

$$\phi(u) := \delta e^u - \gamma u, \;\; \psi(v) := \beta e^v - \alpha v$$

are continuous and

$$\lim_{|u| \to +\infty} \phi(u) = +\infty, \quad \lim_{|v| \to +\infty} \phi(v) = +\infty.$$

As $V(u, v) = \phi(u) + \psi(v)$,

$$\lim_{\|(u,v)\| \to +\infty} V(u, v) = +\infty.$$

Consequently, since $V(x(t), y(t))$ is constant, the solution $(u(t), v(t))$ **cannot** blow up (either in finite or infinite time).

Therefore the solution $(u(t), v(t))$ is global as is the solution in the original variables $(x(t), y(t))$.

Since $(x, y) = F^{-1}(u, v)$ and the domain of F is $]0, +\infty[^2$, $x(t) > 0$ and $y(t) > 0$ for any $t \geq 0$.