

INTERNAL DYNAMICS

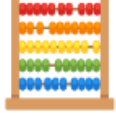


Sébastien Boisgérault

CONTROL ENGINEERING WITH PYTHON

-  Documents (GitHub)
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-  Mines ParisTech, PSL University

SYMBOLS

	Code		Worked Example
	Graph		Exercise
	Definition		Numerical Method
	Theorem		Analytical Method
	Remark		Theory
i	Information		Hint
	Warning		Solution



IMPORTS

```
from numpy import *  
from numpy.linalg import *  
from scipy.linalg import *  
from matplotlib.pyplot import *  
from mpl_toolkits.mplot3d import *  
from scipy.integrate import solve_ivp
```



STREAMPLOT HELPER

```
def Q(f, xs, ys):  
    X, Y = meshgrid(xs, ys)  
    v = vectorize  
    fx = v(lambda x, y: f([x, y])[0])  
    fy = v(lambda x, y: f([x, y])[1])  
    return X, Y, fx(X, Y), fy(X, Y)
```



We are interested in the behavior of the solution to

$$\dot{x} = Ax, \quad x(0) = x_0 \in \mathbb{R}^n$$

First, we study some elementary systems in this class.

SCALAR CASE, REAL-VALUED

$$\dot{x} = ax$$

$$a \in \mathbb{R}, \quad x(0) = x_0 \in \mathbb{R}.$$

 **Solution:**

$$x(t) = e^{at}x_0$$

 **Proof:**

$$\frac{d}{dt}e^{at}x_0 = ae^{at}x_0 = ax(t)$$

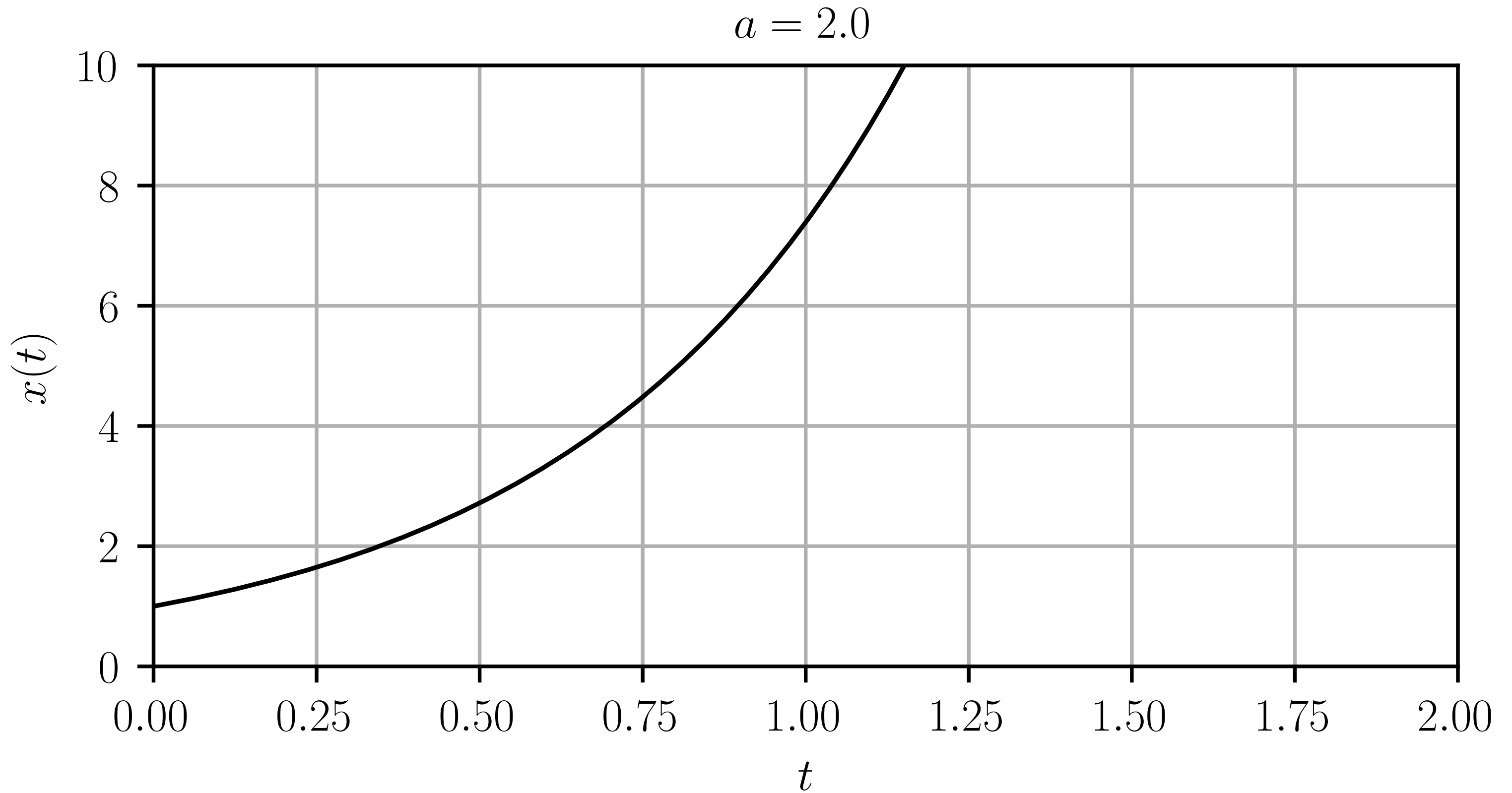
and

$$x(0) = e^{a \times 0}x_0 = x_0.$$



TRAJECTORY

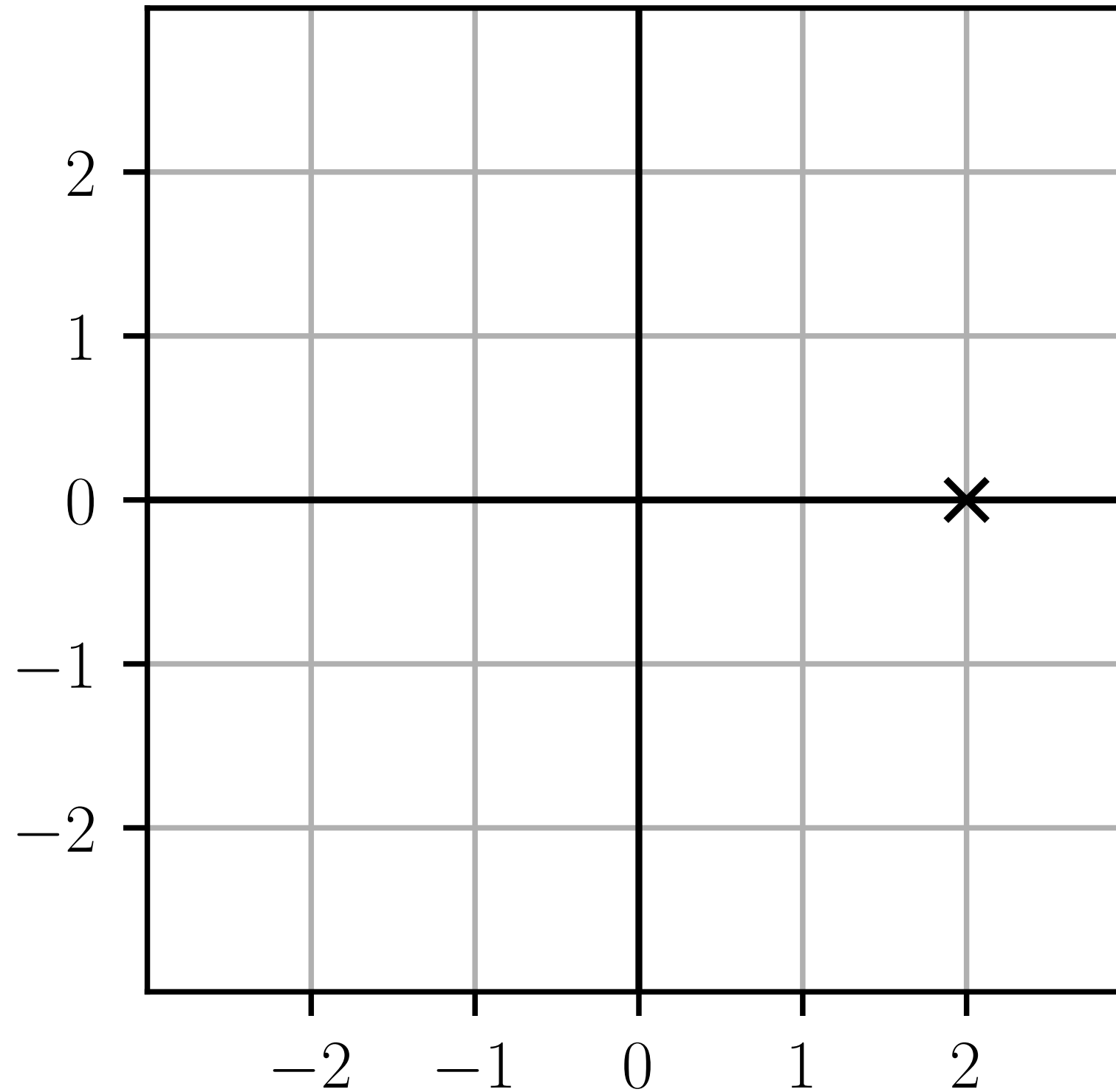
```
a = 2.0; x0 = 1.0
figure()
t = linspace(0.0, 3.0, 1000)
plot(t, exp(a*t)*x0, "k")
xlabel("$t$"); ylabel("$x(t)$"); title(f"$a={a}$")
grid(); axis([0.0, 2.0, 0.0, 10.0])
```





```
figure()
plot(real(a), imag(a), "x", color="k")
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$"); grid(True)
```

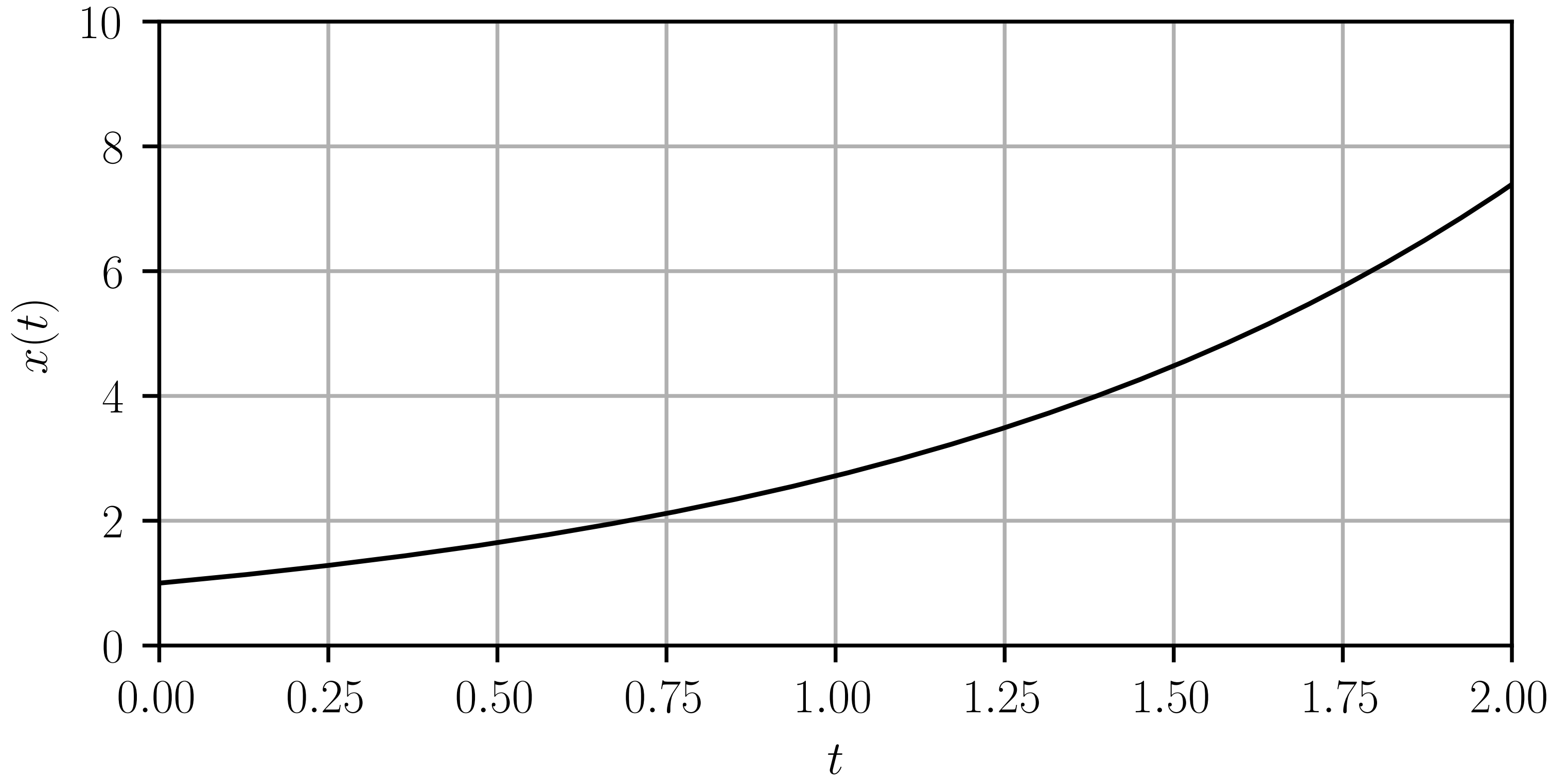
$$a = 2.0$$





```
a = 1.0; x0 = 1.0
figure()
t = linspace(0.0, 3.0, 1000)
plot(t, exp(a*t)*x0, "k")
xlabel("$t$"); ylabel("$x(t)$"); title(f"$a={a}$")
grid(); axis([0.0, 2.0, 0.0, 10.0])
```

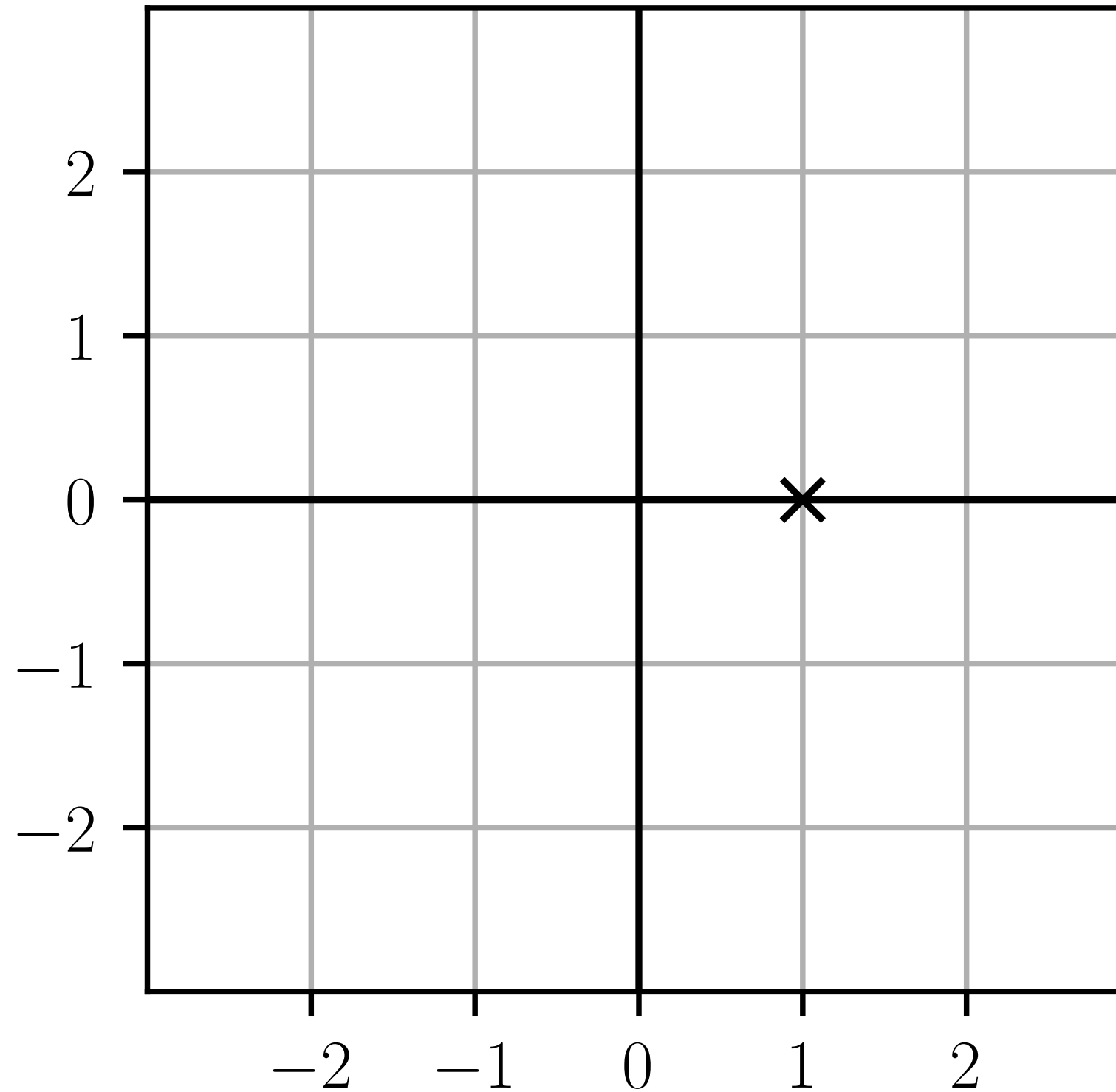
$$a = 1.0$$





```
figure()
plot(real(a), imag(a), "x", color="k")
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$"); grid(True)
```

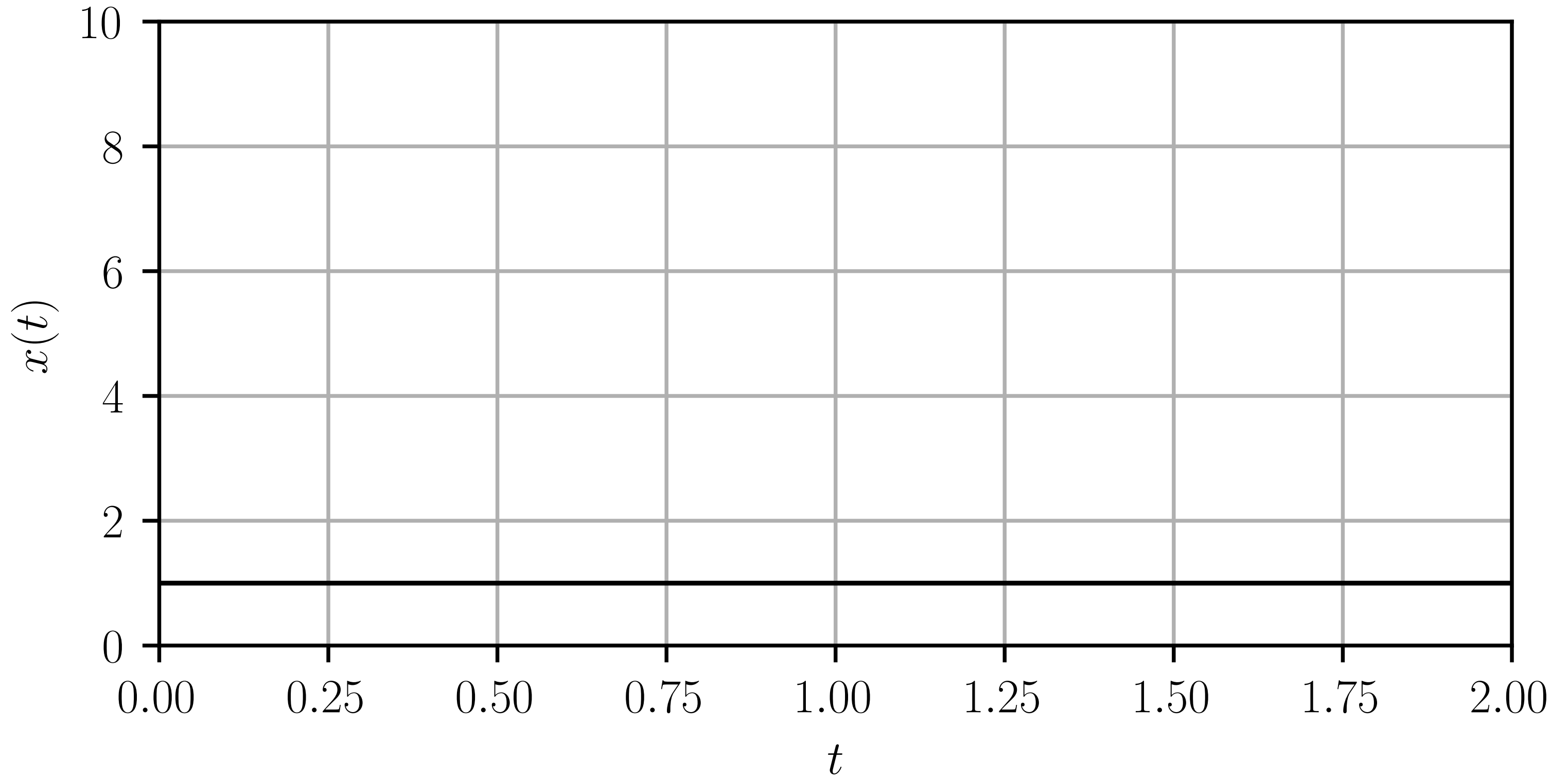

$$a = 1.0$$





```
a = 0.0; x0 = 1.0
figure()
t = linspace(0.0, 3.0, 1000)
plot(t, exp(a*t)*x0, "k")
xlabel("$t$"); ylabel("$x(t)$"); title(f"$a={a}$")
grid(); axis([0.0, 2.0, 0.0, 10.0])
```

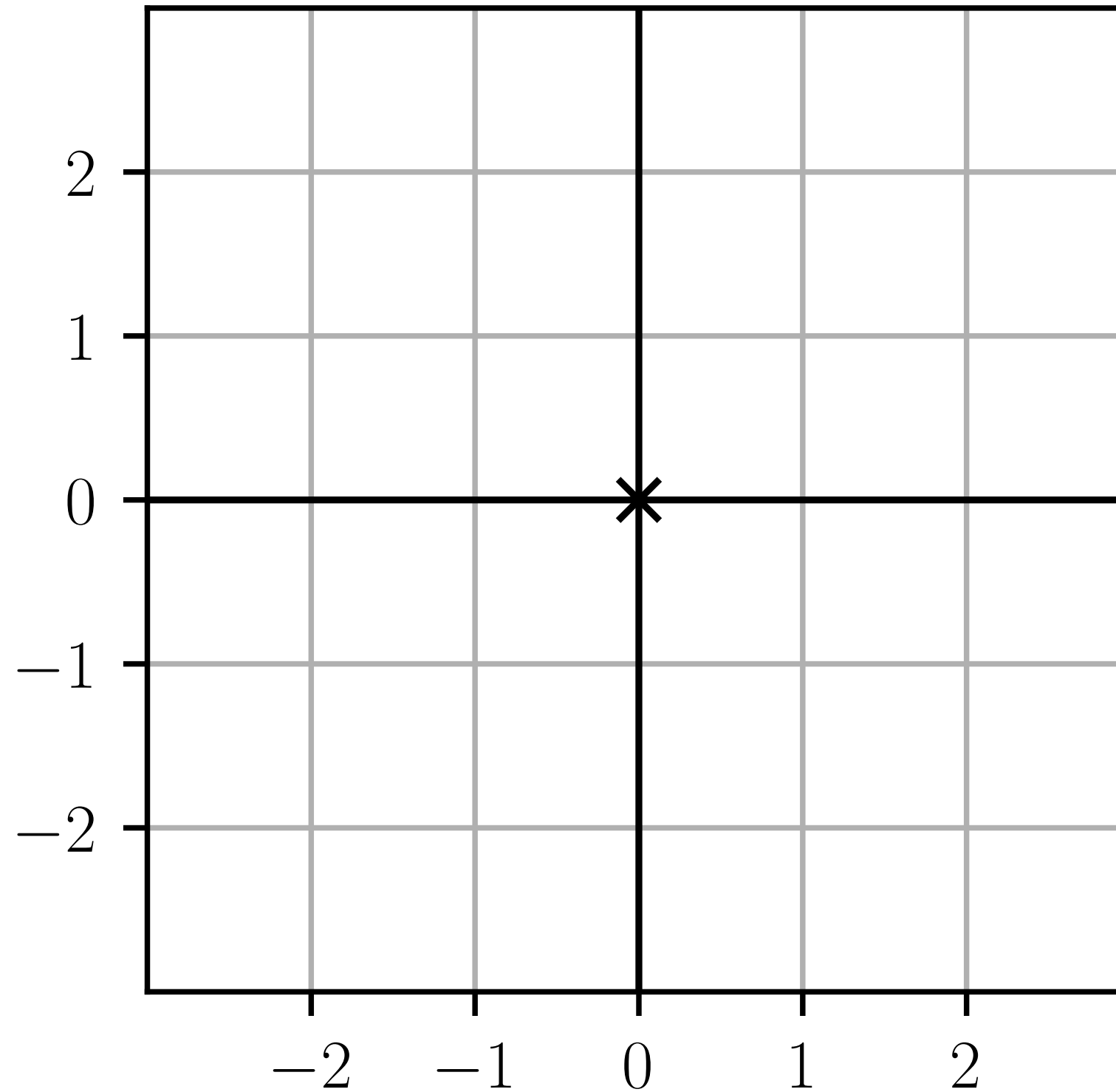
$$a = 0.0$$





```
figure()
plot(real(a), imag(a), "x", color="k")
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$"); grid(True)
```

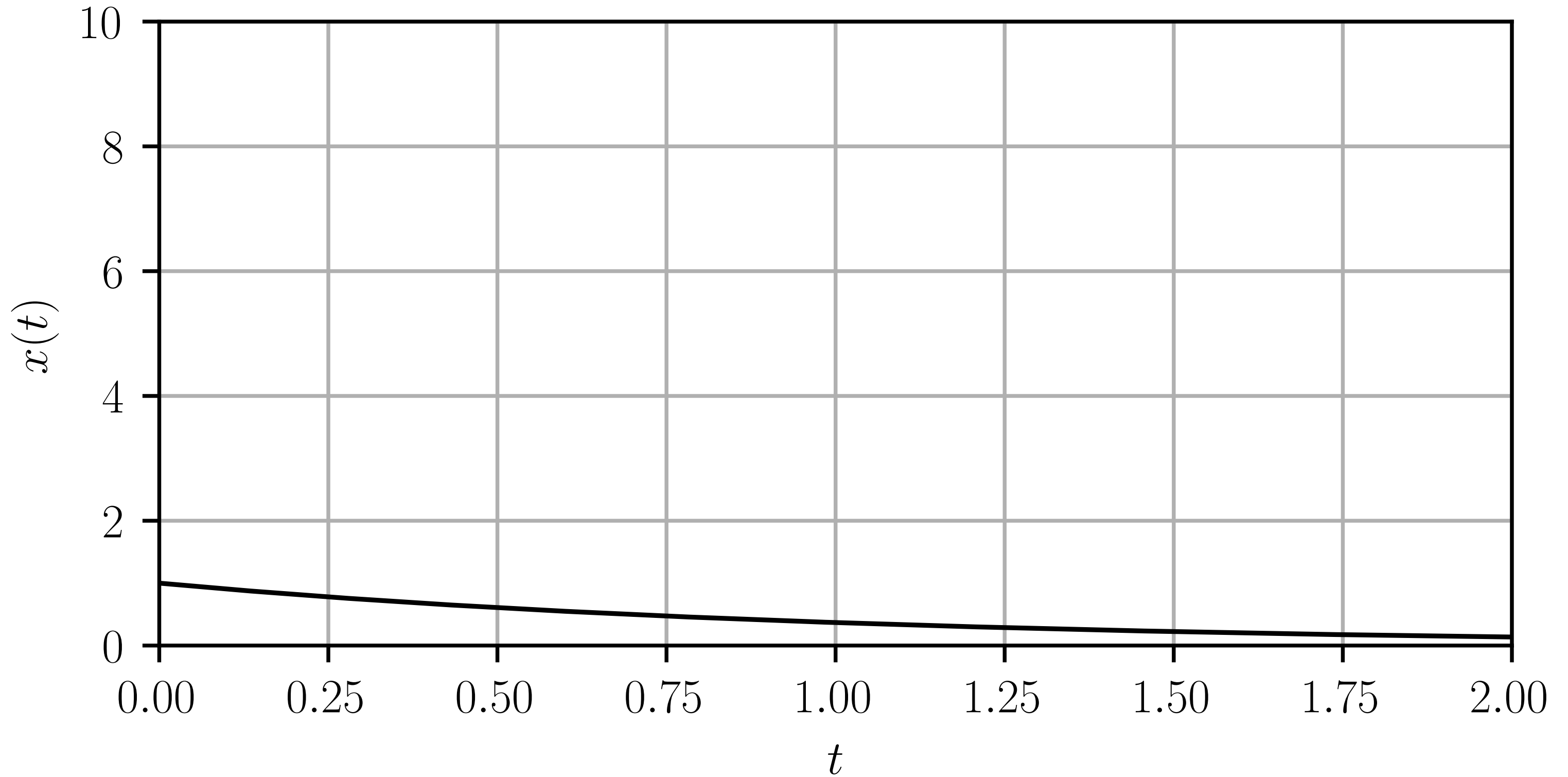
$$a = 0.0$$





```
a = -1.0; x0 = 1.0
figure()
t = linspace(0.0, 3.0, 1000)
plot(t, exp(a*t)*x0, "k")
xlabel("$t$"); ylabel("$x(t)$"); title(f"$a={a}$")
grid(); axis([0.0, 2.0, 0.0, 10.0])
```

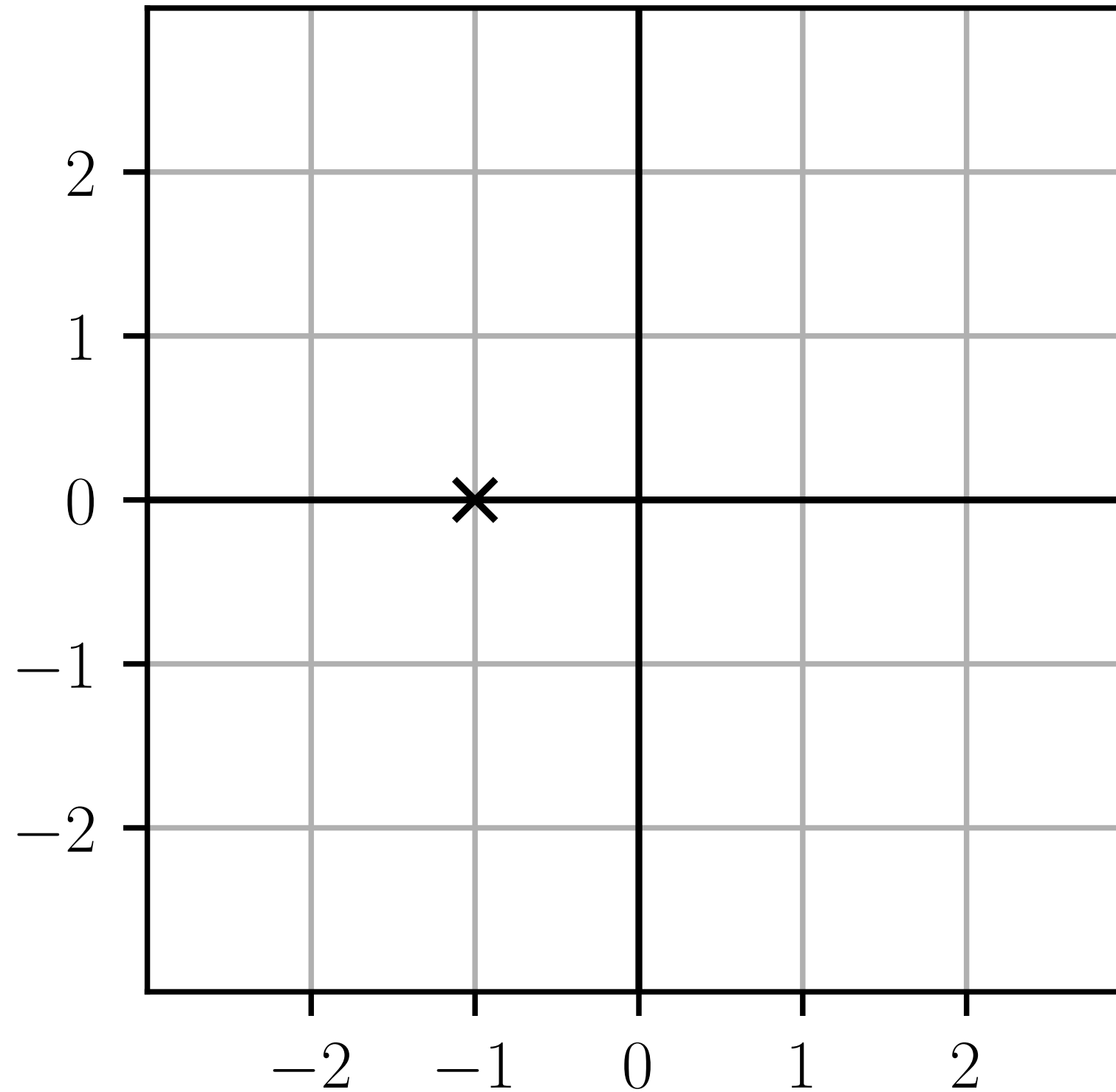
$$a = -1.0$$





```
figure()
plot(real(a), imag(a), "x", color="k")
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$"); grid(True)
```

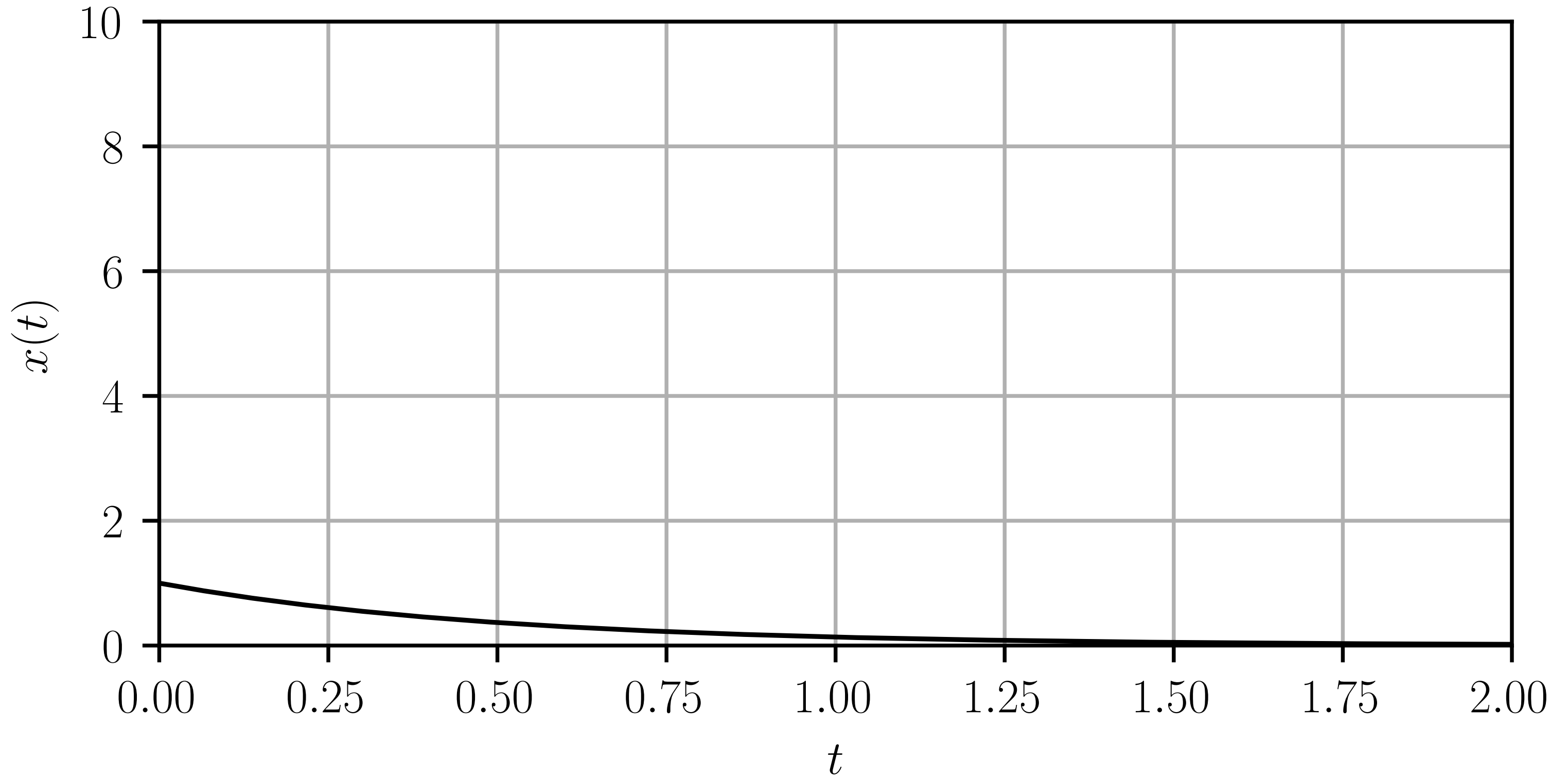

$$a = -1.0$$





```
a = -2.0; x0 = 1.0
figure()
t = linspace(0.0, 3.0, 1000)
plot(t, exp(a*t)*x0, "k")
xlabel("$t$"); ylabel("$x(t)$"); title(f"$a={a}$")
grid(); axis([0.0, 2.0, 0.0, 10.0])
```

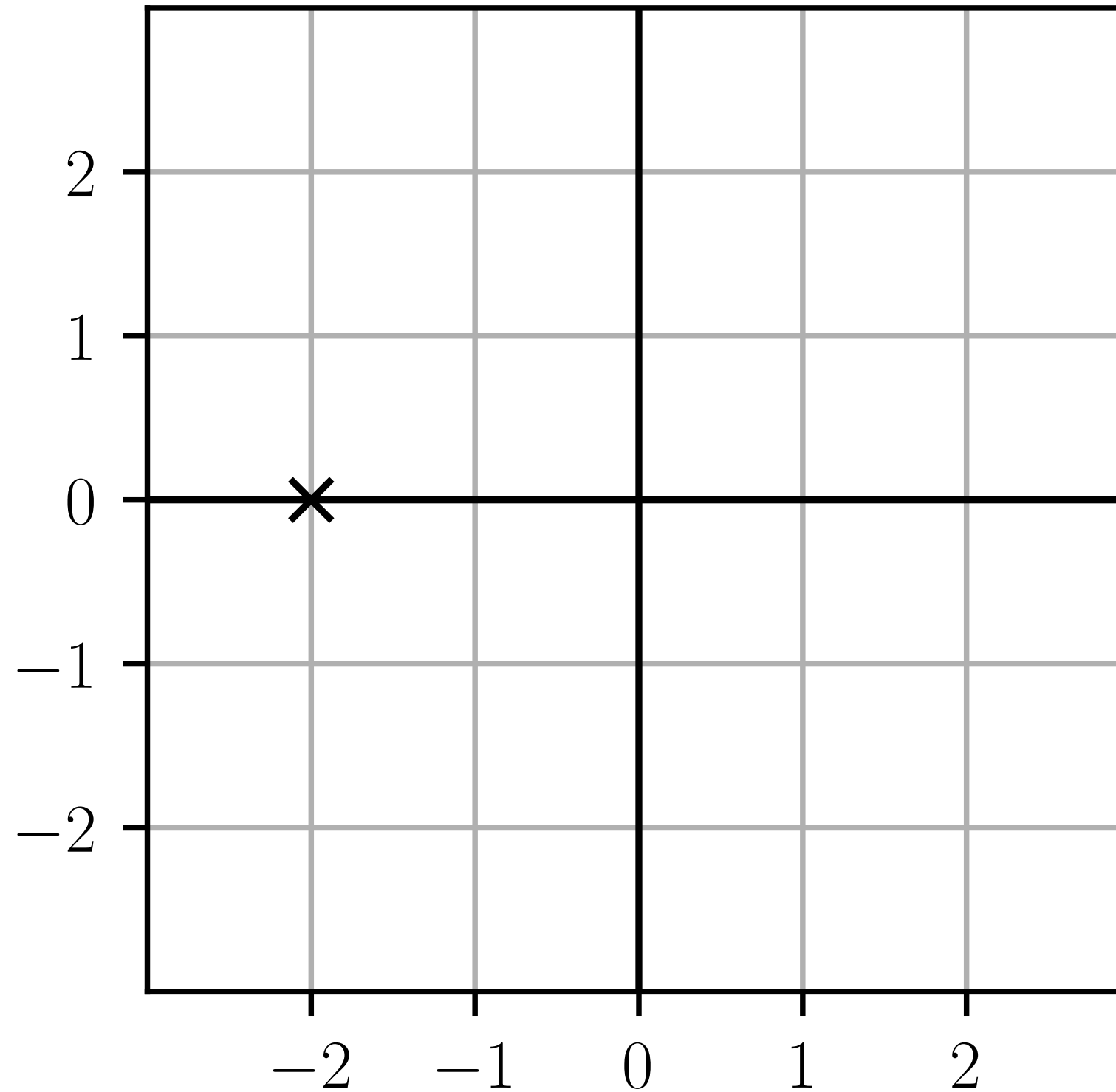
$$a = -2.0$$





```
figure()
plot(real(a), imag(a), "x", color="k")
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$"); grid(True)
```

$$a = -2.0$$





ANALYSIS

The origin is globally asymptotically stable when

$$a < 0.0$$

i.e. a is in the open left-hand plane.

Let the **time constant** τ be

$$\tau := 1/|a|.$$

When the system is asymptotically stable,

$$x(t) = e^{-t/\tau} x_0.$$

QUANTITATIVE CONVERGENCE

τ controls the speed of convergence to the origin:

time t	distance to the origin $ x(t) $
0	$ x(0) $
τ	$\simeq (1/3) x(0) $
3τ	$\simeq (5/100) x(0) $
$\vdots$	$\vdots$
$+\infty$	0

VECTOR CASE, DIAGONAL, REAL- VALUED

$$\dot{x}_1 = a_1 x_1, \quad x_1(0) = x_{10}$$

$$\dot{x}_2 = a_2 x_2, \quad x_2(0) = x_{20}$$

i.e.

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

 **Solution:** by linearity

$$x(t) = e^{a_1 t} \begin{bmatrix} x_{10} \\ 0 \end{bmatrix} + e^{a_2 t} \begin{bmatrix} 0 \\ x_{20} \end{bmatrix}$$



```
a1 = -1.0; a2 = 2.0; x10 = x20 = 1.0
```

```
figure()
```

```
t = linspace(0.0, 3.0, 1000)
```

```
x1 = exp(a1*t)*x10; x2 = exp(a2*t)*x20
```

```
xn = sqrt(x1**2 + x2**2)
```

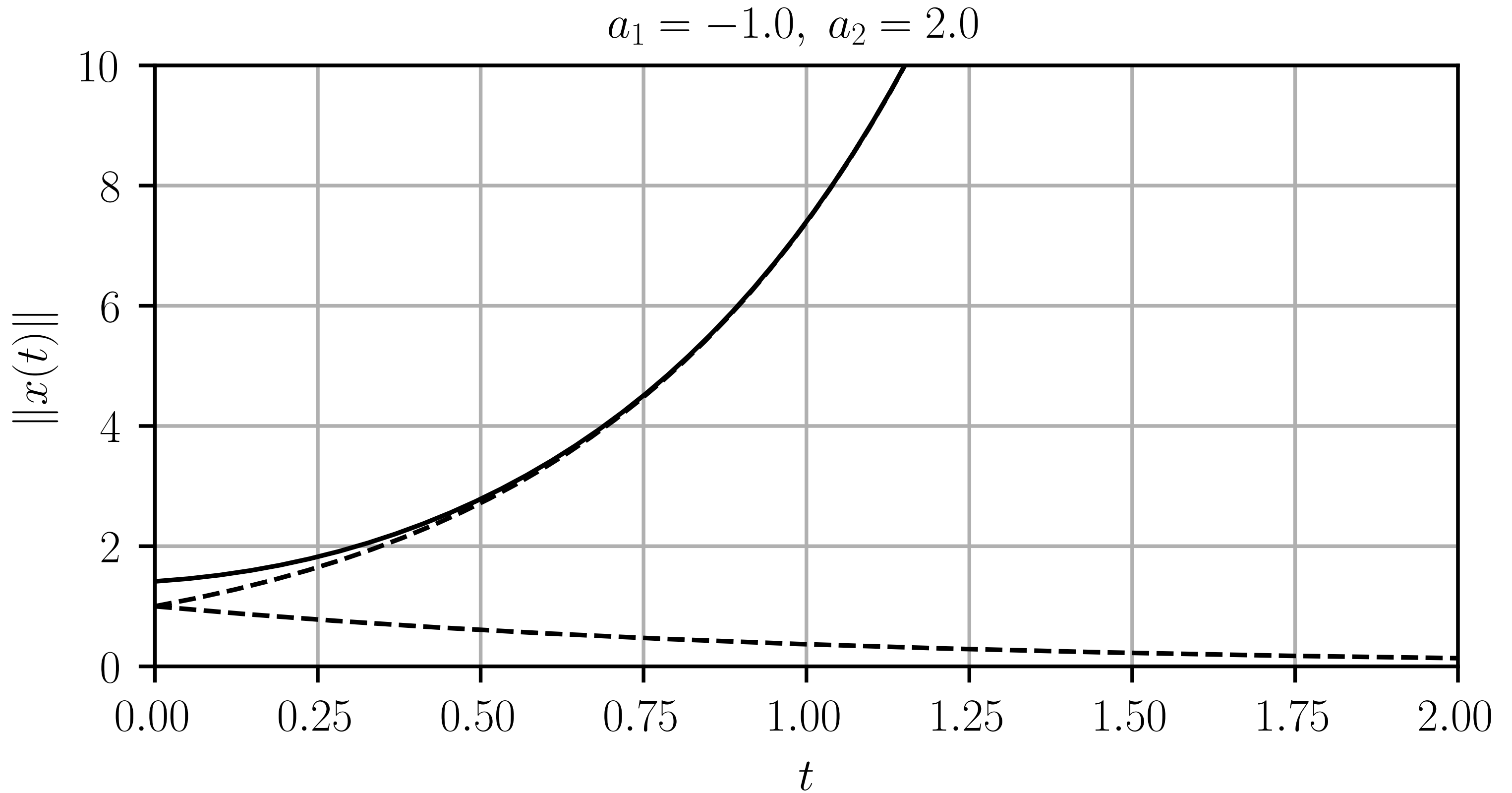
```
plot(t, xn , "k")
```

```
plot(t, x1, "k--")
```

```
plot(t, x2 , "k--")
```

```
xlabel("$t$"); ylabel("$\\|x(t)\\|$"); title(f"$a_1={a1}$, \\; $a_2$")
```

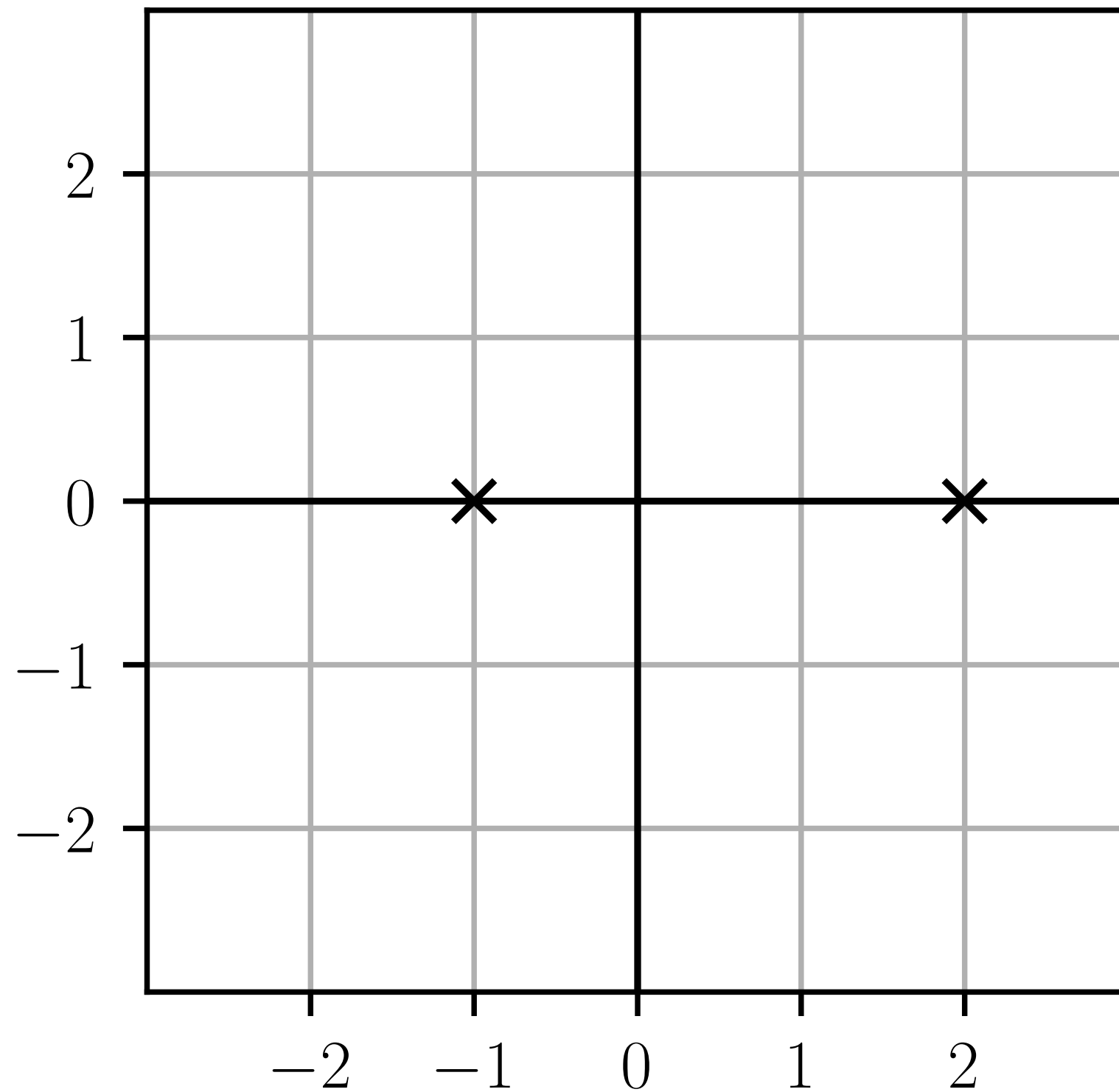
```
grid(); axis([0.0, 2.0, 0.0, 10.0])
```





```
figure()
plot(real(a1), imag(a1), "x", color="k")
plot(real(a2), imag(a2), "x", color="k")
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a_1={a1}, \; a_2={a2}$")
grid(True)
```

$$a_1 = -1.0, \quad a_2 = 2.0$$





```
a1 = -1.0; a2 = -2.0; x10 = x20 = 1.0
```

```
figure()
```

```
t = linspace(0.0, 3.0, 1000)
```

```
x1 = exp(a1*t)*x10; x2 = exp(a2*t)*x20
```

```
xn = sqrt(x1**2 + x2**2)
```

```
plot(t, xn , "k")
```

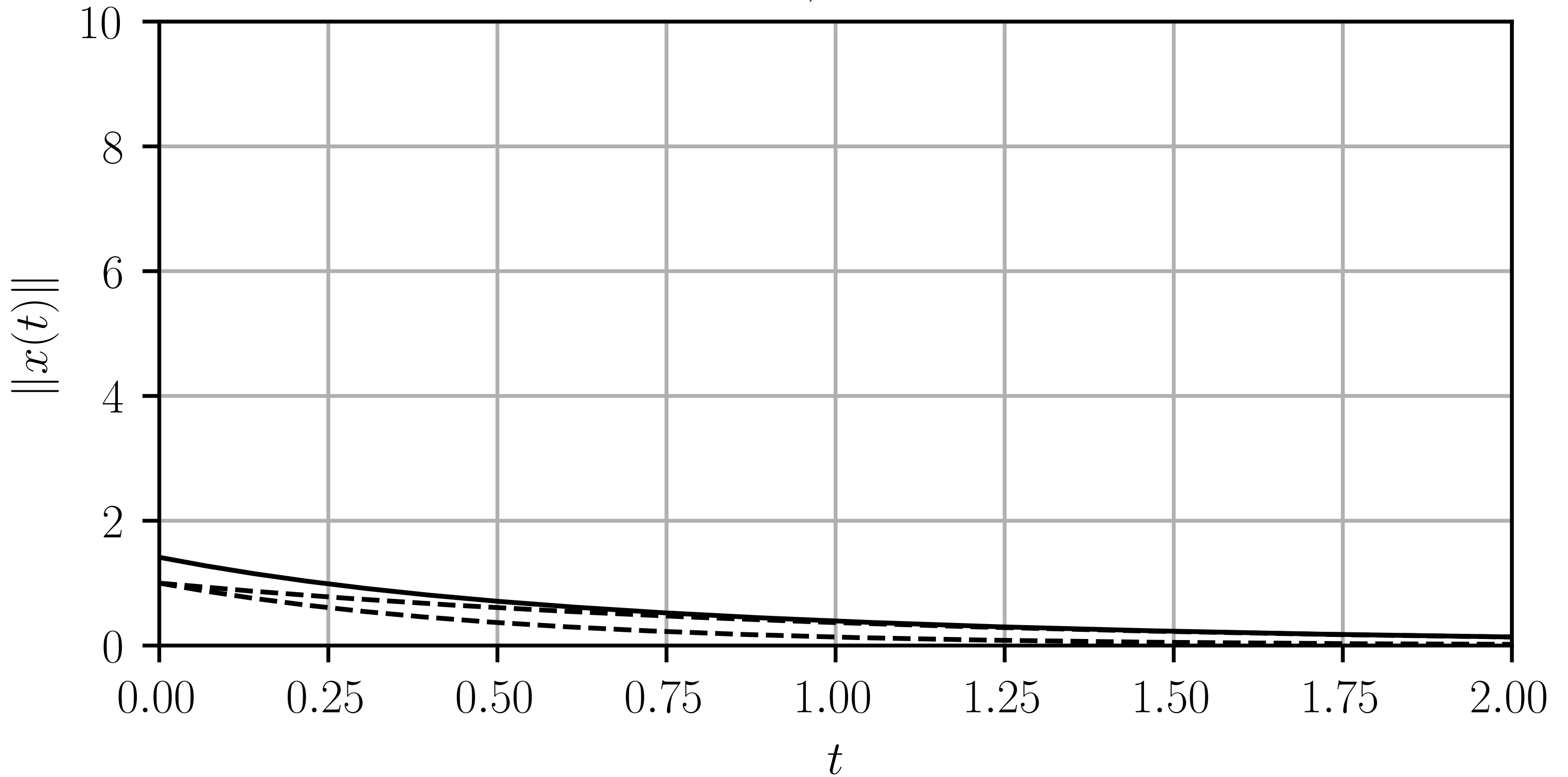
```
plot(t, x1, "k--")
```

```
plot(t, x2 , "k--")
```

```
xlabel("$t$"); ylabel("$\\|x(t)\\|$"); title(f"$a_1={a1}$, \\; $a_2$")
```

```
grid(); axis([0.0, 2.0, 0.0, 10.0])
```

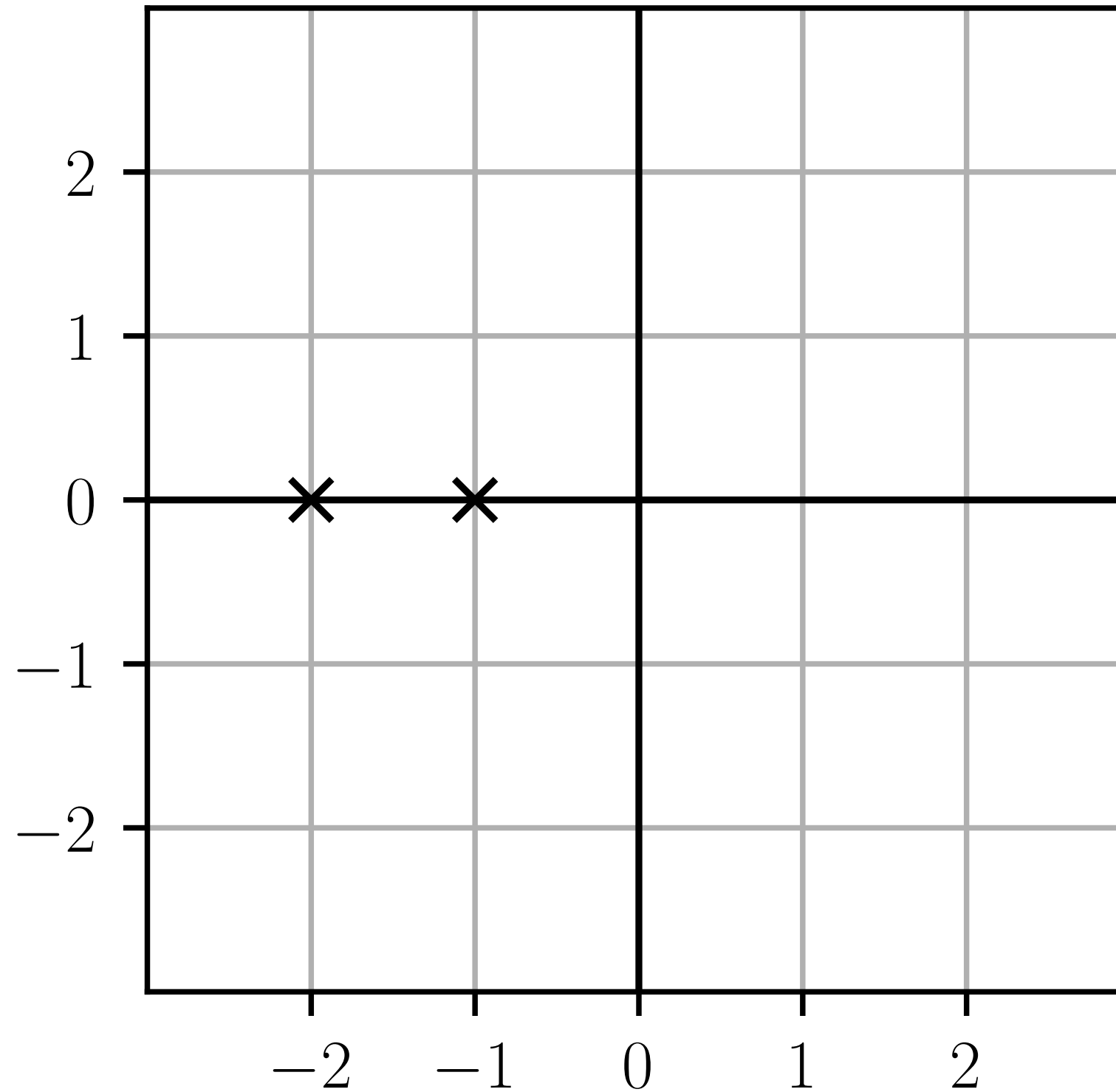
$$a_1 = -1.0, \quad a_2 = -2.0$$





```
figure()
plot(real(a1), imag(a1), "x", color="k")
plot(real(a2), imag(a2), "x", color="k")
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a_1={a1}, \; a_2={a2}$")
grid(True)
```


$$a_1 = -1.0, \quad a_2 = -2.0$$





ANALYSIS

-  The rightmost a_i determines the asymptotic behavior,
-  The origin is globally asymptotically stable if and only if
every a_i is in the open left-hand plane.

SCALAR CASE, COMPLEX-VALUED

$$\dot{x} = ax$$

$$a \in \mathbb{C}, x(0) = x_0 \in \mathbb{C}.$$



Solution: formally, the same old solution

$$x(t) = e^{at} x_0$$

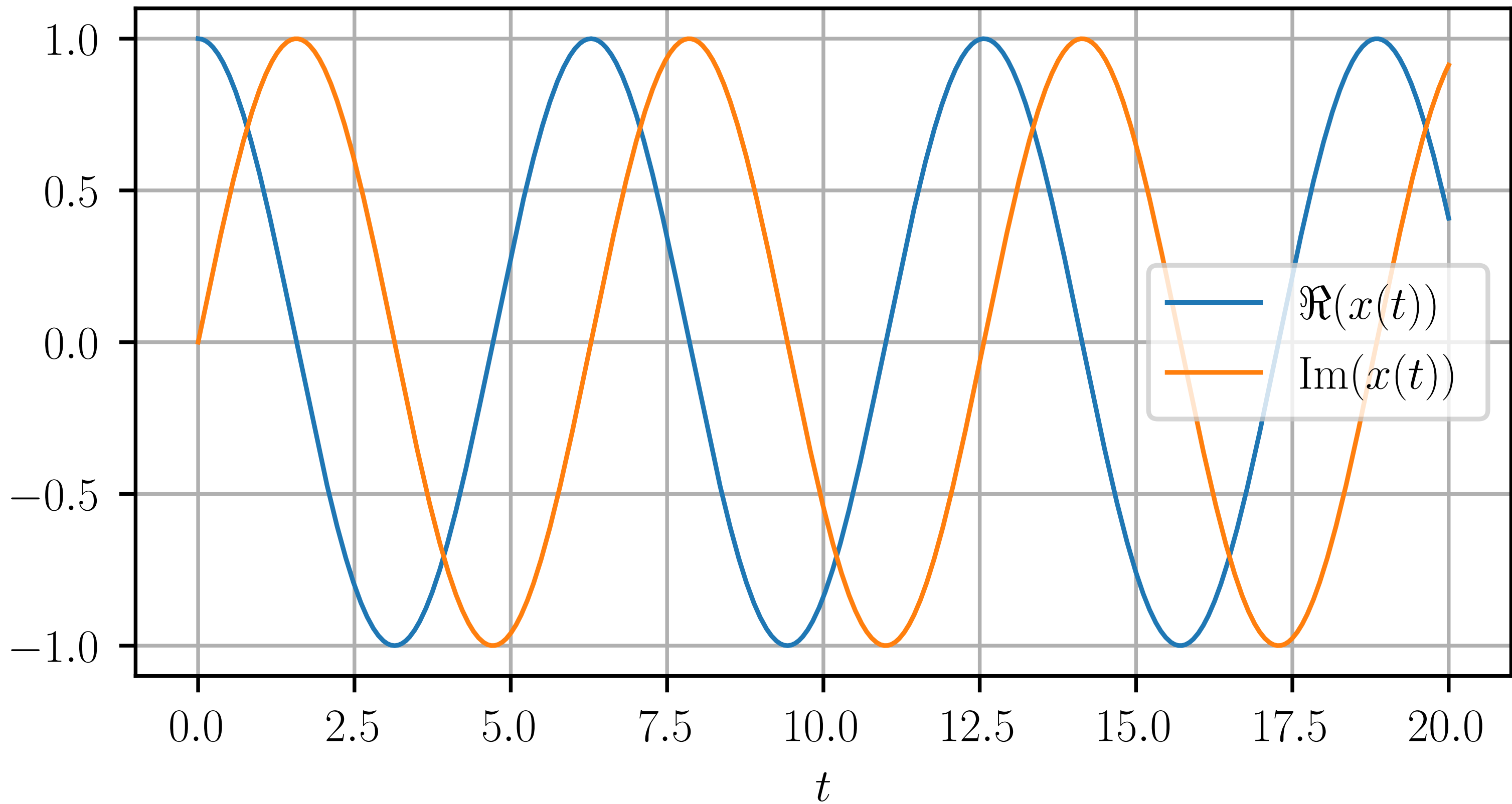
But now, $x(t) \in \mathbb{C}$:

if $a = \sigma + i\omega$ and $x_0 = |x_0|e^{i\angle x_0}$

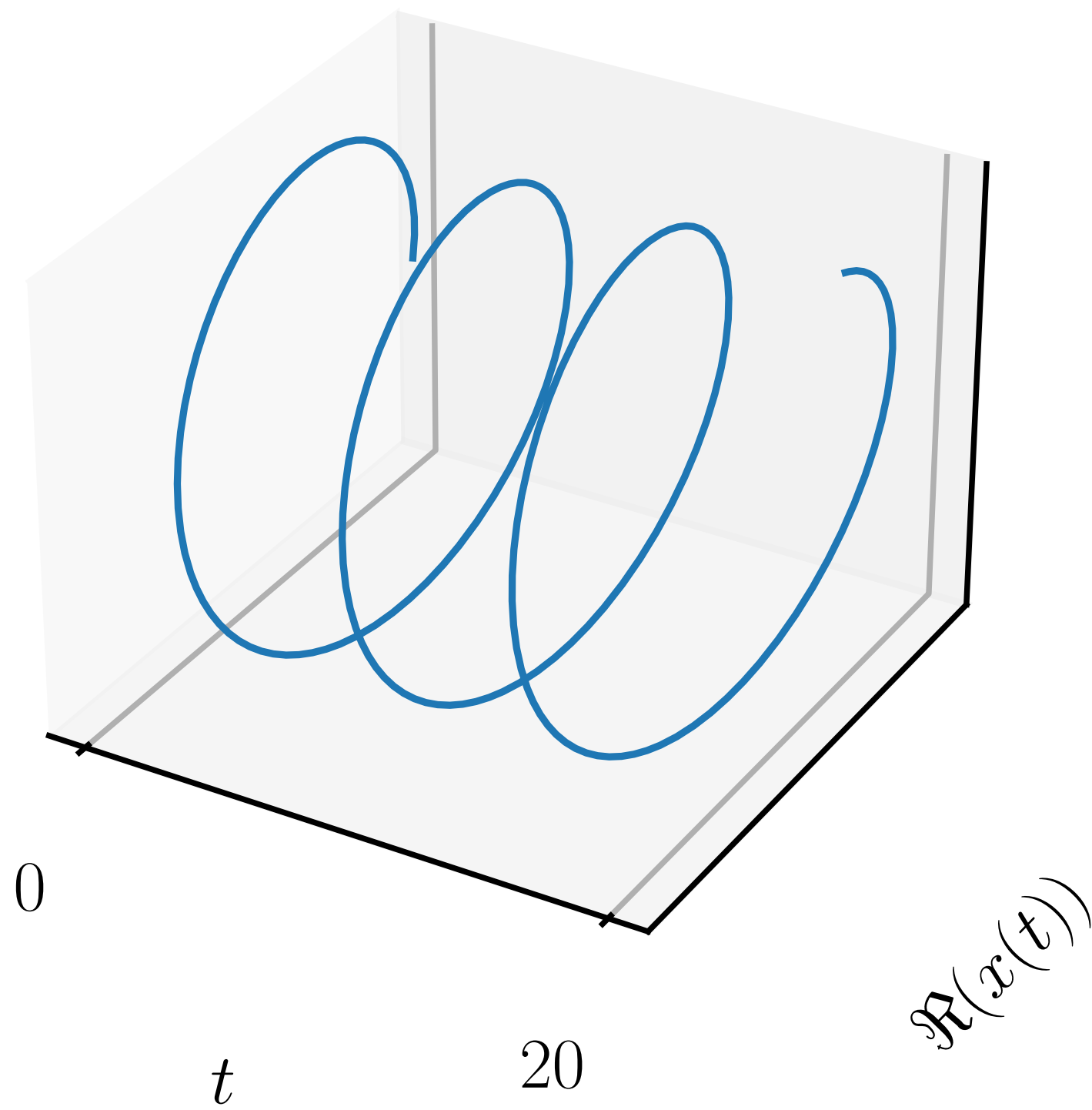
$$|x(t)| = |x_0|e^{\sigma t} \quad \text{and} \quad \angle x(t) = \angle x_0 + \omega t.$$



```
a = 1.0j; x0=1.0
figure()
t = linspace(0.0, 20.0, 1000)
plot(t, real(exp(a*t)*x0), label="$\text{Re}(x(t))$")
plot(t, imag(exp(a*t)*x0), label="$\text{Im}(x(t))$")
xlabel("$t$")
legend(); grid()
```



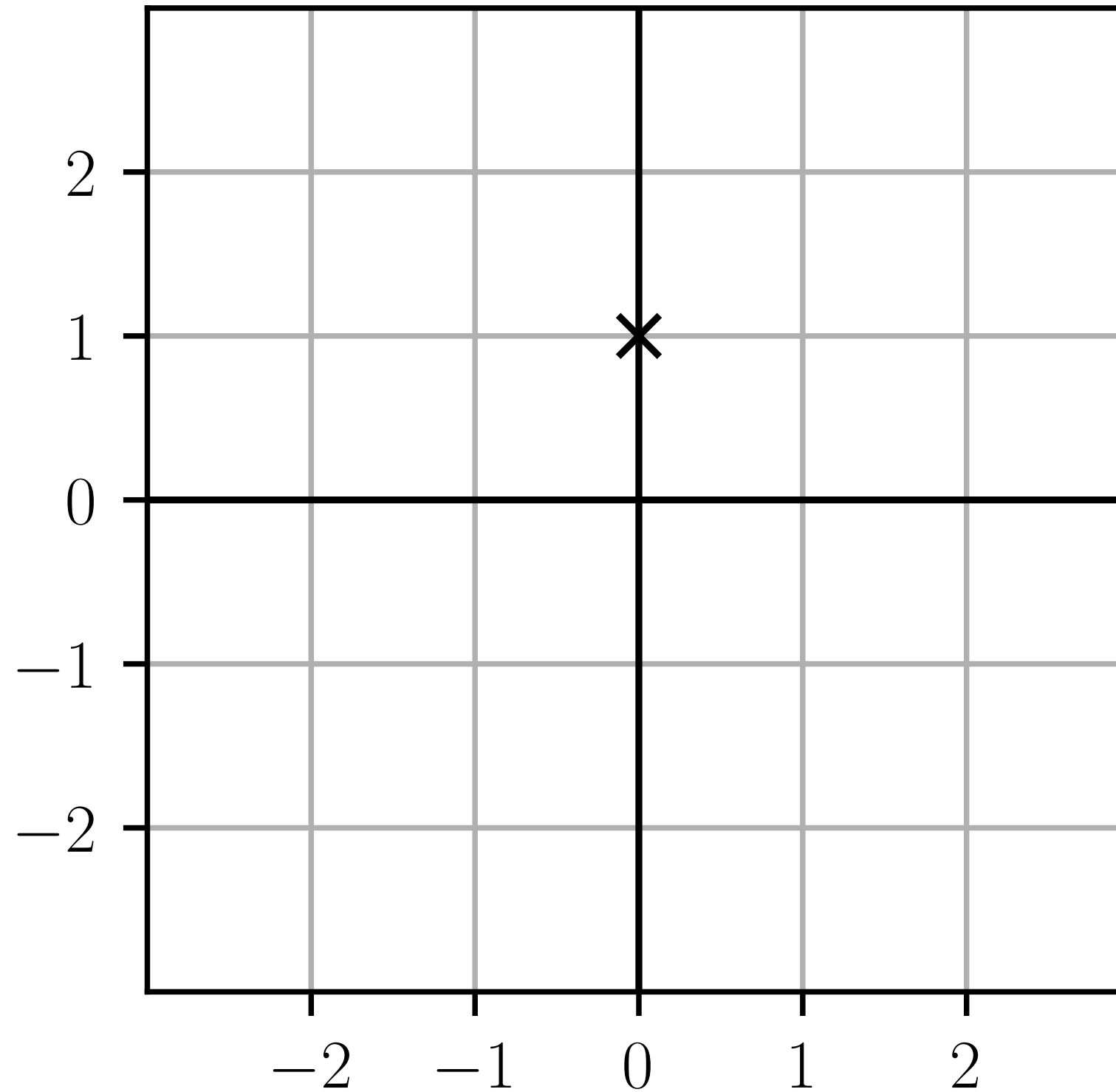
```
fig = figure()
ax = fig.add_subplot(111, projection="3d")
zticks = ax.set_zticks
ax.plot(t, real(exp(a*t)*x0), imag(exp(a*t)*x0))
xticks([0.0, 20.0]); yticks([]); zticks([])
ax.set_xlabel("$t$")
ax.set_ylabel("$\text{Re}(x(t))$")
ax.set_zlabel("$\mathrm{Im}(x(t))$")
```



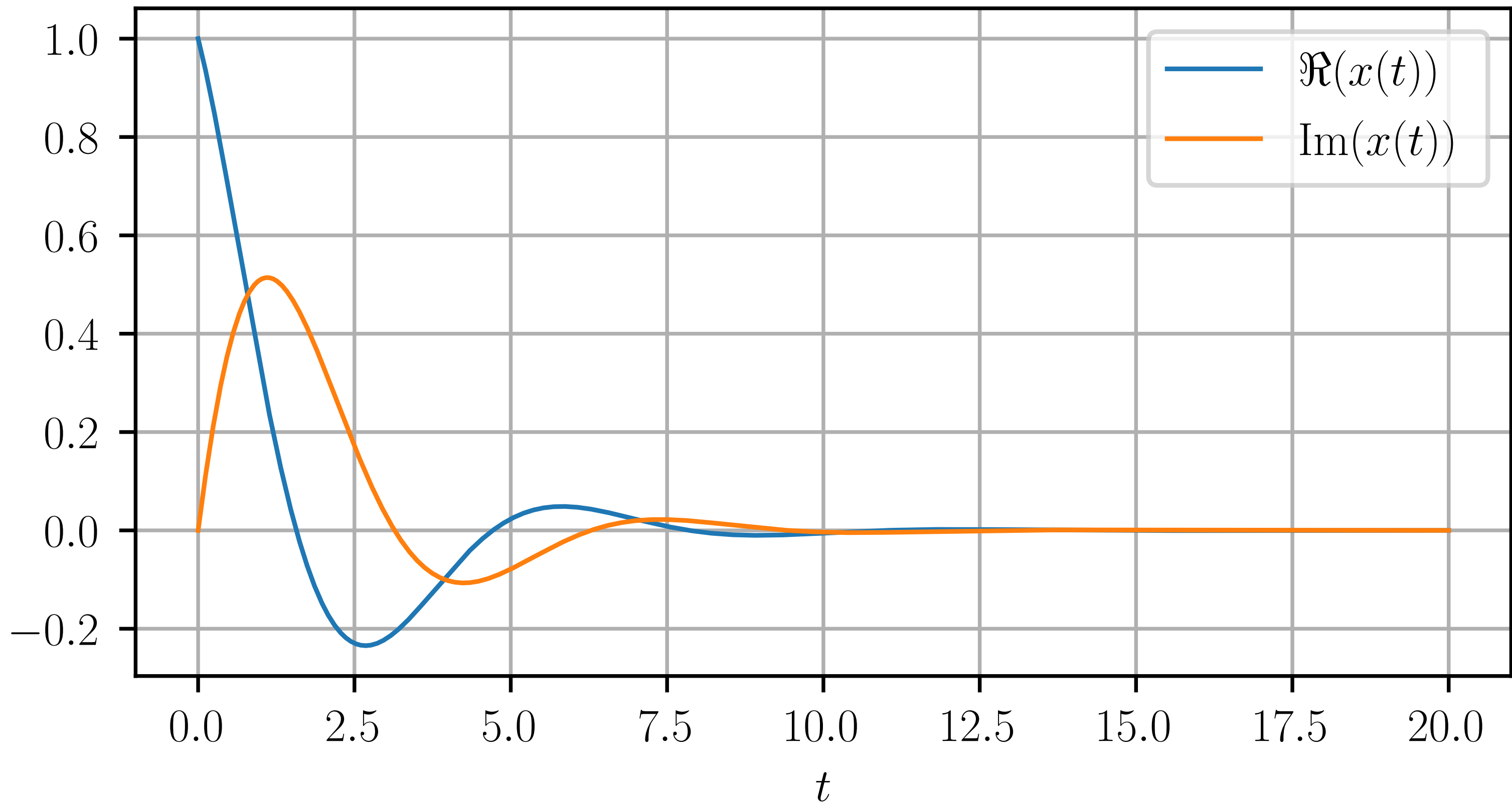
```
figure()
plot(real(a), imag(a), "x", color="k")
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$"); grid(True)
```

$$a = 1j$$



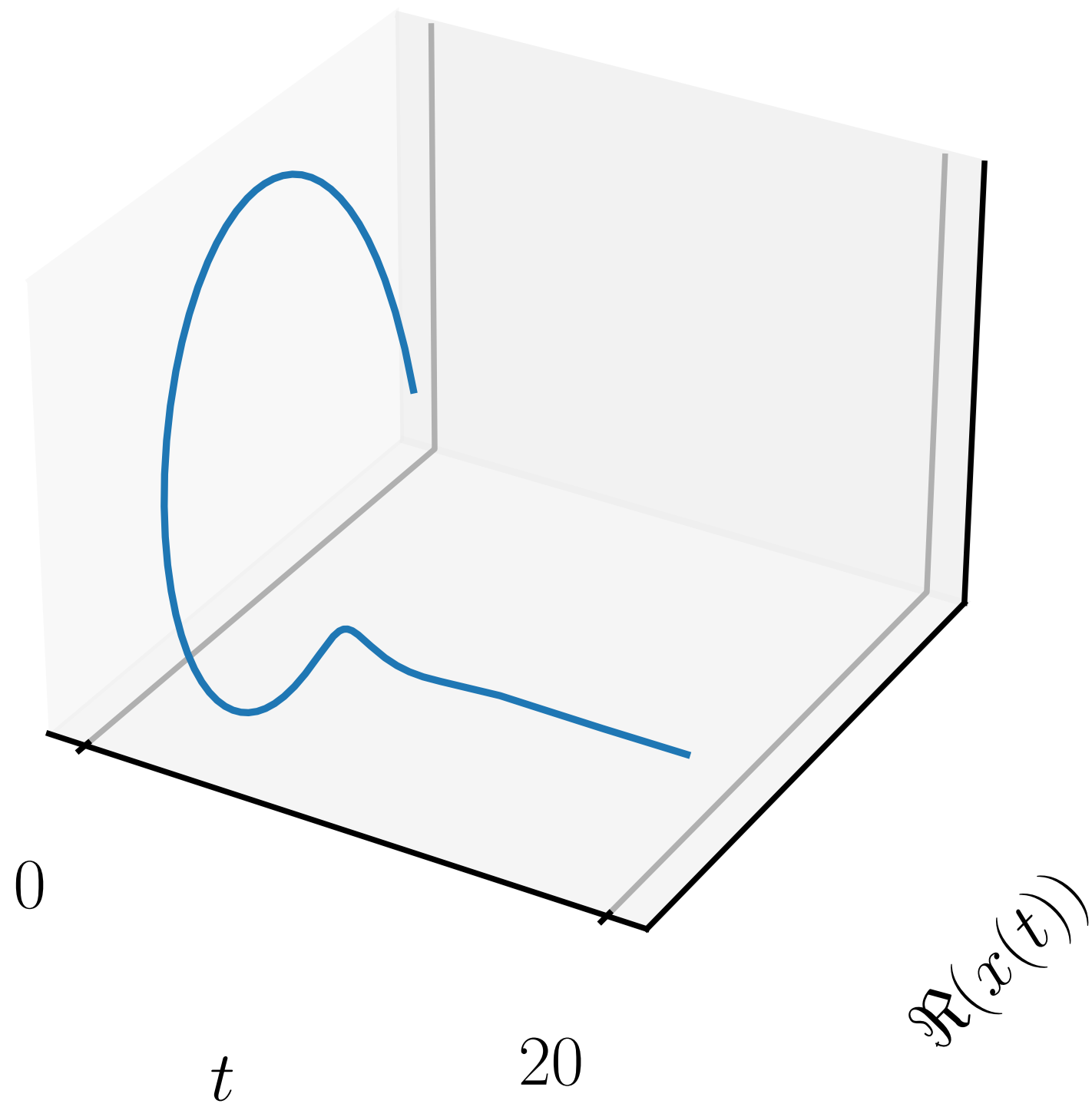


```
a = -0.5 + 1.0j; x0=1.0
figure()
t = linspace(0.0, 20.0, 1000)
plot(t, real(exp(a*t)*x0), label="$\text{Re}(x(t))$")
plot(t, imag(exp(a*t)*x0), label="$\text{Im}(x(t))$")
xlabel("$t$")
legend(); grid()
```





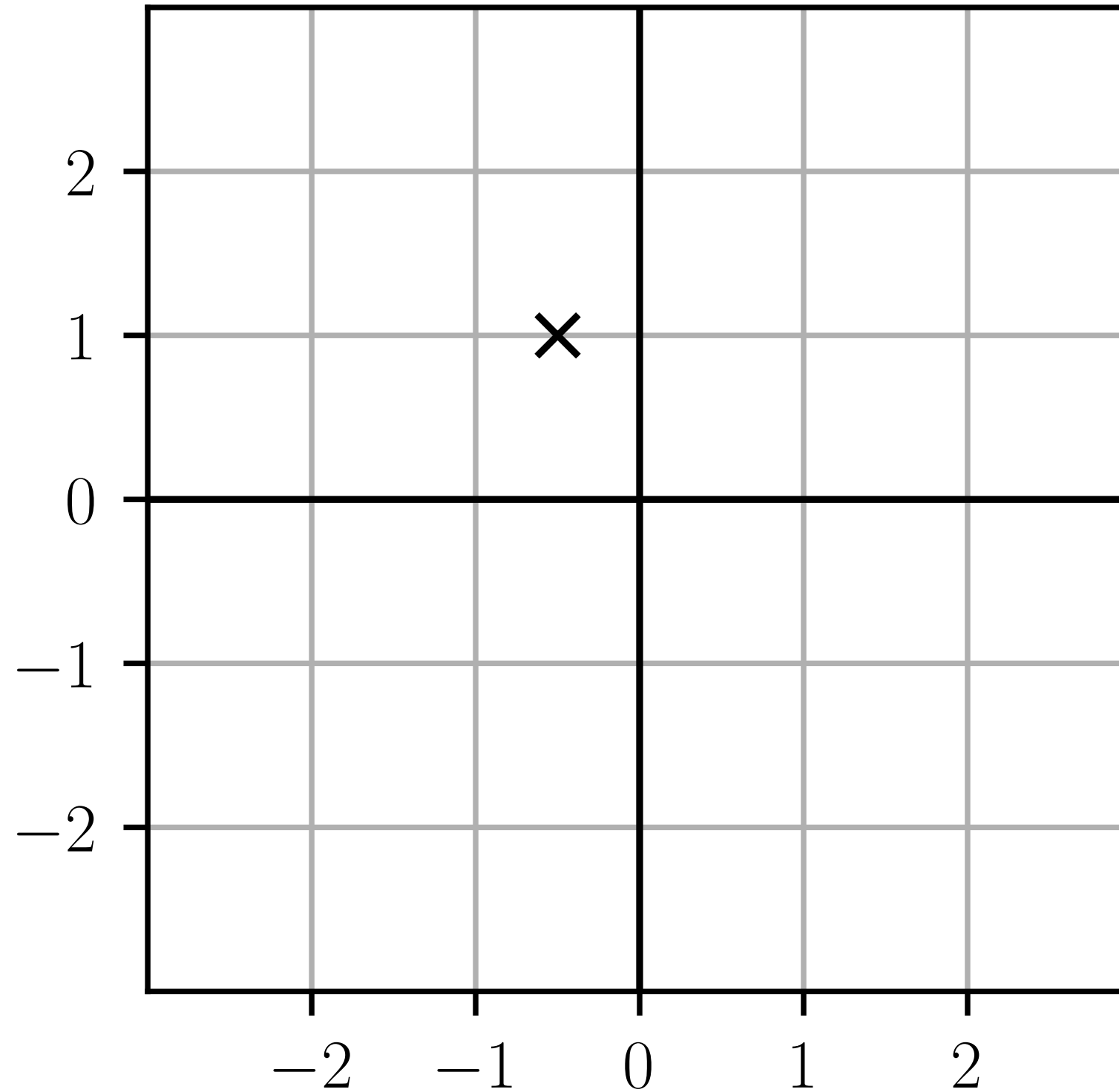
```
fig = figure()
ax = fig.add_subplot(111, projection="3d")
zticks = ax.set_zticks
ax.plot(t, real(exp(a*t)*x0), imag(exp(a*t)*x0))
xticks([0.0, 20.0]); yticks([]); zticks([])
ax.set_xlabel("$t$")
ax.set_ylabel("$\text{Re}(x(t))$")
ax.set_zlabel("$\mathrm{Im}(x(t))$")
```





```
figure()
plot(real(a), imag(a), "x", color="k")
gca().set_aspect(1.0)
xlim(-3,3); ylim(-3,3);
plot([-3,3], [0,0], "k")
plot([0, 0], [-3, 3], "k")
xticks([-2,-1,0,1,2]); yticks([-2,-1,0,1,2])
title(f"$a={a}$")
grid(True)
```


$$a = (-0.5 + 1j)$$





ANALYSIS

- The origin is globally asymptotically stable iff a is in the open left-hand plane: $\Re(a) < 0$.
- If $a =: \sigma + i\omega$,
 - $\tau = 1/|\sigma|$ is the **time constant**.
 - ω the **rotational frequency** of the oscillations.



EXPONENTIAL MATRIX

If $M \in \mathbb{C}^{n \times n}$, its exponential is defined as:

$$e^M = \sum_{k=0}^{+\infty} \frac{M^k}{k!} \in \mathbb{C}^{n \times n}$$



The exponential of a matrix M is **not** the matrix with elements $e^{M_{ij}}$ (the elementwise exponential).

- 🐍 elementwise exponential: **exp** (numpy module),
- 🐍 exponential: **expm** (scipy.linalg module).



EXPONENTIAL MATRIX

Let

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

1. 

Compute the exponential of M .

 **Hint:**

$$\cosh x := \frac{e^x + e^{-x}}{2}, \quad \sinh x := \frac{e^x - e^{-x}}{2}.$$

2. 🐍 💻 🔬

Compute numerically:

- `exp(M)` (numpy)
- `expm(M)` (scipy.linalg)

and check the results consistency.



EXPONENTIAL MATRIX

1.

We have

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and hence for any $j \in \mathbb{N}$,

$$M^{2j+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M^{2j} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$$\begin{aligned}
e^M &= \sum_{k=0}^{+\infty} \frac{M^k}{k!} \\
&= \left(\sum_{j=0}^{+\infty} \frac{1}{(2j)!} \right) I + \left(\sum_{j=0}^{+\infty} \frac{1}{(2j+1)!} \right) M \\
&= \left(\sum_{k=0}^{+\infty} \frac{1^k + (-1)^k}{2(k!)} \right) I + \left(\sum_{k=0}^{+\infty} \frac{1^k - (-1)^k}{2(k!)} \right) M \\
&= (\cosh 1)I + (\sinh 1)M
\end{aligned}$$

Thus,

$$e^M = \begin{bmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{bmatrix}.$$

2.

```
>>> M = [[0.0, 1.0], [1.0, 0.0]]
```

```
>>> exp(M)
array([[1.          , 2.71828183],
       [2.71828183, 1.          ]])
```

```
>>> expm(M)
array([[1.54308063, 1.17520119],
       [1.17520119, 1.54308063]])
```

These results are consistent:

```
>>> array([[exp(0.0), exp(1.0)],  
...       [exp(1.0), exp(0.0)]])  
array([[1.          , 2.71828183],  
       [2.71828183, 1.          ]])
```

```
>>> array([[cosh(1.0), sinh(1.0)],  
...       [sinh(1.0), cosh(1.0)]])  
array([[1.54308063, 1.17520119],  
       [1.17520119, 1.54308063]])
```



Note that

$$\begin{aligned}\frac{d}{dt}e^{At} &= \frac{d}{dt} \sum_{n=0}^{+\infty} \frac{A^n}{n!} t^n \\ &= \sum_{n=1}^{+\infty} \frac{A^n}{(n-1)!} t^{n-1} \\ &= A \sum_{n=1}^{+\infty} \frac{A^{n-1}}{(n-1)!} t^{n-1} = Ae^{At}\end{aligned}$$

Thus, for any $A \in \mathbb{C}^{n \times n}$ and $x_0 \in \mathbb{C}^n$,

$$\frac{d}{dt}(e^{At}x_0) = A(e^{At}x_0)$$



INTERNAL DYNAMICS

The solution of

$$\dot{x} = Ax \text{ and } x(0) = x_0$$

is

$$x(t) = e^{At}x_0.$$

G.A.S. $\Leftrightarrow$ L.A.

 For any dynamical system, if the origin is a globally asymptotically stable equilibrium, then it is a locally attractive equilibrium.

 **For linear systems**, the converse result also holds.

 Let's prove this!

1. 🧠

Show that for any linear system $\dot{x} = Ax$, if the origin is locally attractive, then it is also globally attractive.

2.

Show that linear system $\dot{x} = Ax$, if the origin is globally attractive, then it is also globally asymptotically stable.

 **Hint:** Consider the solutions $e_k(t) := e^{At}e_k$ associated to $e_k(0) = e_k$ where $(e_1, \dots, e_n)$ is the canonical basis of the state space.



G.A.S. $\Leftrightarrow$ L.A.

1.

If the origin is locally attractive, then there is a $\varepsilon > 0$ such that for any $x_0 \in \mathbb{R}^n$ such that $\|x_0\| \leq \varepsilon$,

$$\lim_{t \rightarrow +\infty} e^{At} x_0 = 0.$$

Now, let any $x_0 \in \mathbb{R}^n$. Since the norm of $\varepsilon x_0 / \|x_0\|$ is ε , and by linearity of e^{At} , we obtain

$$\begin{aligned}\lim_{t \rightarrow +\infty} e^{At} x_0 &= \lim_{t \rightarrow +\infty} e^{At} \left(\frac{\|x_0\|}{\varepsilon} \varepsilon \frac{x_0}{\|x_0\|} \right) \\ &= \frac{\|x_0\|}{\varepsilon} \lim_{t \rightarrow +\infty} e^{At} \left(\varepsilon \frac{x_0}{\|x_0\|} \right) \\ &= 0.\end{aligned}$$

Thus the origin is globally attractive.

2.

Let X_0 be a bounded set of $\mathbb{R}^n$. Since

$$x_0 = \sum_{k=1}^n x_{0k} e_k,$$

the solution $x(t)$ of $\dot{x} = Ax$, $x(0) = x_0$ satisfies

$$x(t) = e^{At} x_0 = e^{At} \left(\sum_{k=1}^n x_{0k} e_k \right) = \sum_{k=1}^n x_{0k} e^{At} e_k.$$

$$\begin{aligned}
\|x(t)\| &= \left\| \sum_{k=1}^n x_{0k} e^{At} e_k \right\| \\
&\leq \sum_{k=1}^n |x_{0k}| \|e^{At} e_k\| \\
&= \sum_{k=1}^n |x_{0k}| \|e_k(t)\| \\
&\leq \left(\sum_{k=1}^n |x_{0k}| \right) \max_{k=1, \dots, n} \|e_k(t)\|
\end{aligned}$$

Since X_0 is bounded, there is a $\alpha > 0$ such that for any $x_0 = (x_{01}, \dots, x_{0n})$ in X_0 ,

$$\|x_0\|_1 := \sum_{k=1}^n |x_{0k}| \leq \alpha.$$

Since for every $k = 1, \dots, n$, $\lim_{t \rightarrow +\infty} \|e_k(t)\| = 0$,

$$\lim_{t \rightarrow +\infty} \max_{k=1, \dots, n} \|e_k(t)\| = 0.$$

Finally

$$\begin{aligned} \|x(t, x_0)\| &\leq \left(\sum_{k=1}^n |x_{0k}| \right) \max_{k=1, \dots, n} \|e_k(t)\| \\ &\leq \alpha \max_{k=1, \dots, n} \|e_k(t)\| \end{aligned}$$

Thus $\|x(t, x_0)\| \rightarrow 0$ when $t \rightarrow \infty$, uniformly w.r.t. $x_0 \in X_0$. In other words, the origin is globally asymptotically stable.



EIGENVALUE & EIGENVECTOR

Let $A \in \mathbb{C}^n$. If $x \neq 0 \in \mathbb{C}^n$, $s \in \mathbb{C}$ and

$$Ax = sx$$

x is an **eigenvector** of A , s is an **eigenvalue** of A .

The **spectrum** of A is the set of its eigenvalues.

It is characterized by:

$$\sigma(A) := \{s \in \mathbb{C} \mid \det(sI - A) = 0\}.$$



MODES & POLES

Consider the system $\dot{x} = Ax$.

- a **mode** of the system is an eigenvector of A ,
- a **pole** of the system is an eigenvalue of A .



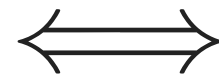
STABILITY CRITERIA

Let $A \in \mathbb{C}^{n \times n}$.

The origin of $\dot{x} = Ax$ is globally asymptotically stable



all eigenvalues of A have a negative real part.



$$\max\{\Re s \mid s \in \sigma(A)\} < 0.$$

WHY DOES THIS CRITERIA WORK?

Assume that:

- A is diagonalizable.

( very likely unless A has some special structure.)

Let $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$.

There is an invertible matrix $P \in \mathbb{C}^{n \times n}$ such that

$$P^{-1}AP = \text{diag}(\lambda_1, \dots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \lambda_n \end{bmatrix}$$

Thus, if $y = P^{-1}x$, $\dot{x} = Ax$ is equivalent to

$$\begin{cases} \dot{y}_1 &= \lambda_1 y_1 \\ \dot{y}_2 &= \lambda_2 y_2 \\ \vdots &= \vdots \\ \dot{y}_n &= \lambda_n y_n \end{cases}$$

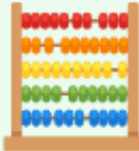
The system is G.A.S. iff each component of the system is, which holds iff $\Re \lambda_i < 0$ for each i .



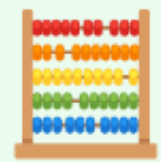
SPRING-MASS SYSTEM

Consider the scalar ODE

$$\ddot{x} + kx = 0, \quad \text{with } k > 0$$

1. 

Represent this system as a first-order ODE.

2.  

Is this system asymptotically stable?

3.

Do the solutions have oscillatory components?

Find the set of associated rotational frequencies.

4.

Same set of questions (1., 2., 3.) for

$$\ddot{x} + b\dot{x} + kx = 0$$

when $b > 0$.



SPRING-MASS SYSTEM

1. 

$$\frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = A \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

2. 

We have

$$\max\{\Re s \mid s \in \sigma(A)\} = 0,$$

hence the system is not globally asymptotically stable.

3.

Since

$$\det(sI - A) = \det \begin{pmatrix} s & -1 \\ k & s \end{pmatrix} = s^2 + k,$$

the spectrum of A is

$$\sigma(A) = \{s \in \mathbb{C} \mid \det(sI - A) = 0\} = \{i\sqrt{k}, -i\sqrt{k}\}.$$

The system poles are $\pm i\sqrt{k}$.

The general solution $x(t)$ can be decomposed as

$$x(t) = x_+ e^{i\sqrt{k}t} + x_- e^{-i\sqrt{k}t}.$$

Thus the components of $x(t)$ oscillate at the rotational frequency

$$\omega = \sqrt{k}.$$

4. 

$$\frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = A \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

$$\det(sI - A) = \det \begin{pmatrix} s & -1 \\ k & s + b \end{pmatrix} = s^2 + bs + k,$$

Let $\Delta := b^2 - 4k$. If $b \geq 2\sqrt{k}$, then $\Delta \geq 0$ and

$$\sigma(A) = \left\{ \frac{-b + \sqrt{\Delta}}{2}, \frac{-b - \sqrt{\Delta}}{2} \right\}.$$

Otherwise,

$$\sigma(A) = \left\{ \frac{-b + i\sqrt{-\Delta}}{2}, \frac{-b - i\sqrt{-\Delta}}{2} \right\}.$$

Thus, if $b \geq 2\sqrt{k}$,

$$\max\{\Re s \mid s \in \sigma(A)\} = \frac{-b + \sqrt{b^2 - 4k}}{2} < 0$$

and otherwise

$$\max\{\Re s \mid s \in \sigma(A)\} = -\frac{b}{2} < 0.$$

In each case, the system is globally asymptotically stable.

If $b \geq 2\sqrt{k}$, the poles are real-valued; the components of the solution do not oscillate.

If $0 < b < 2\sqrt{k}$, the imaginary part of the poles is

$$\pm \frac{\sqrt{4k - b^2}}{2} = \pm \sqrt{k - (b/2)^2},$$

thus the solution components oscillate at the rotational frequency

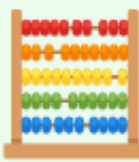
$$\omega = \sqrt{k - (b/2)^2}.$$



INTEGRATOR CHAIN

Consider the system

$$\dot{x} = Jx \quad \text{with} \quad J = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

1. 

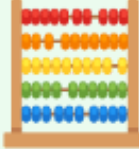
Compute the solution $x(t)$ when

$$x(0) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

2. 🦊 🧮

Compute the solution for an arbitrary $x(0)$

$$x(0) = \begin{bmatrix} x_1(0) \\ \vdots \\ \vdots \\ x_n(0) \end{bmatrix}.$$

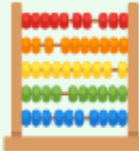
3. 

Same questions for the system

$$\dot{x} = (\lambda I + J)x$$

for some $\lambda \in \mathbb{C}$.

 **Hint:** Find the ODE satisfied by $y(t) := x(t)e^{-\lambda t}$.

4. 

Is the system asymptotically stable ?

5. 🧠

Why does the stability analysis of this system matter ?



INTEGRATOR CHAIN

1.

Let $x = (x_1, \dots, x_n)$.

The ODE $\dot{x} = Jx$ is equivalent to:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\vdots \quad \quad \vdots \quad \quad \vdots$$

$$\dot{x}_{n-1} = x_n$$

$$\dot{x}_n = 0.$$

When $x(0) = (0, \dots, 0, 1)$,

- $\dot{x}_n = 0$ yields $x_n(t) = 1$, then
- $\dot{x}_{n-1} = x_n$ yields $x_{n-1}(t) = t$,
- ...
- $\dot{x}_k = x_{k+1}$ yields

$$x_k(t) = \frac{t^{n-k}}{(n-k)!}.$$

To summarize:

$$x(t) = \begin{bmatrix} t^{n-1}/(n-1)! \\ \vdots \\ t^{n-1-k}/(n-1-k)! \\ \vdots \\ t \\ 1 \end{bmatrix}$$

2.

We note that

$$x(0) = x_1(0) \begin{bmatrix} 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \cdots + x_{n-1}(0) \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} + x_n(0) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} .$$

Similarly to the previous question, we find that:

$$x(0) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow x(t) = \begin{bmatrix} t^{n-2}/(n-2)! \\ \vdots \\ t \\ 1 \\ 0 \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow x(t) = \begin{bmatrix} t^{n-3}/(n-3)! \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

And more generally, by linearity:

$$x(t) = \begin{bmatrix} x_1(0) + \cdots + x_{n-1}(0) \frac{t^{n-2}}{(n-2)!} + x_n(0) \frac{t^{n-1}}{(n-1)!} \\ \vdots \\ x_{n-2}(0) + x_{n-1}(0)t + x_n(0) \frac{t^2}{2} \\ x_{n-1}(0) + x_n(0)t \\ x_n(0) \end{bmatrix}$$

3.

If $\dot{x}(t) = (\lambda I + J)x(t)$ and $y(t) = x(t)e^{-\lambda t}$, then

$$\begin{aligned}\dot{y}(t) &= \dot{x}(t)e^{-\lambda t} + x(t)(-\lambda e^{-\lambda t}) \\ &= (\lambda I + J)x(t)e^{-\lambda t} - \lambda Ix(t)e^{-\lambda t} \\ &= Jx(t)e^{-\lambda t} \\ &= Jy(t).\end{aligned}$$

Since $y(0) = x(0)e^{-\lambda 0} = x(0)$ we get

$$x(t) = \begin{bmatrix} x_1(0) + \cdots + x_n(0) \frac{t^{n-1}}{(n-1)!} \\ \vdots \\ x_{n-1}(0) + x_n(0)t \\ x_n(0) \end{bmatrix} e^{\lambda t}.$$

4.

The structure of $x(t)$ shows that

- If $\Re \lambda < 0$, then the system is asymptotically stable.
- If $\Re \lambda \geq 0$, then the system is not.

For example when $x(0) = (1, 0, \dots, 0)$, we have

$$x(t) = (1, 0, \dots, 0).$$

5.

Every square complex matrix A , **even if it is not diagonalizable**, can be decomposed into a block-diagonal matrix where each block has the structure $\lambda I + J$.

Thus, the result of the previous question allows to prove the  **Stability Criteria** in the general case.